Paper - I **Evaluation of double and triple integrals** **Practical -1**

Objective Questions

1. An expression for $\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)dydx$ in which the order of integration is reversed is
   (a) $\int_{-1}^{1} \int_{-y^2}^{y^2} f(x,y)dxdy$. (b) $\int_{-1}^{1} \int_{y^2}^{1} f(x,y)dxdy$.
   (c) a sum of two integrals. (d) None of these.
2. $I = \int_0^1 \int_{1-x}^{1+x} xy \ dydx$. Then I is
   (a) Undefined (b) $\int_0^1 \int_0^y xy \ dxdy$. (c) 0 (d) None of these.
3. $I = \int_0^1 \int_{x^2}^{x} xf(y) \ dydx$ where f is continuous function defined on [0, 1]. Then I is
   (a) $\frac{1}{2}\int_0^1 (y - y^2) f(y)dy$ (b) independent of $f(y)$.
   (c) $\frac{1}{2}\int_0^1 (y^2 - y) f(y)dy$ (d) $f(x)$
4. The value of the double integral $\int_{-1}^{1} \int_0^1 e^{x^2} \sin y \, dxdy$ is equal to
   (a) $2\cos 1 \ \int_0^1 e^{x^2} dx$. (b) $-2\cos 1 \ \int_0^1 e^{x^2} dx$ (c) 0 (d) does not exist.
5. The double integral $\int_0^1 \int_0^x x \ dydx$ reduces to
   (a) $\frac{1}{2}\int_0^1 (1-y)dy$ (b) $\int_0^1 \int_0^y x \ dxdy$ (c) $\int_0^1 \int_y^1 x \ dxdy$ (d) $\frac{1}{2}\int_0^1 x \, dx$
6. If $f(x,y) = k, k$ constant and $R = [a,b] \times [c,d]$ then $\int \int_R \ k \, dA$ equals
   (a) $k(b-a)(d-c)$ (b) $k(c-a)(d-b)$ (c) $k(b-c)(d-a)$ (d) data insufficient
7. Let $D = \{(x,y): x^2 + y^2 \le r\}$ and $f(x,y) = x^2 + y$. Then $\int \int_D \ fdA$ lies in between
   (a) $-16\pi \ and \ 4\pi$ (b) $-2 \ and \ 2$ (c) $-8\pi \ and \ 24\pi$ (d) $-4\pi \ and \ 8\pi$
8. The iterated integral $\int_0^2 \int_{x^2}^{2x} (x^2 + y^2) \ dydx$ represents
   (a) The area of the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$
   (b) Volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $x = y/2$ and $x = \sqrt{y}$
   (c) Volume of the solid the lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by $y^2 = x$ and $x = \sqrt{y}$
   (d) None of the above.
9. The volume of the region bounded by $z = x + y, z = 6, x = 0, y = 0, z = 0$ is
   (a) 36 cubic units (b) 30 cubic units (c) 2/6 cubic units (d) None of these.
10. The volume of the solid given by $x^2 + y^2 \le 1$ and $tan^{-1}\frac{y}{x} \le z \le 2\pi$ is
    (a) π (b) π^2 (c) 1 (d) None of these.
11. Let V be the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.
    Let $A = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) \ dydx$, $B = \int_0^4 \int_{y/2}^{\sqrt{y}} \int_0^{x^2+y^2} dzdxdy$. Then
    (a) $V = A$ but $V \neq B$ (b) $V = A = B$ (C) $V \neq A$ but $V = B$ (d) $V \neq A, V \neq B$

12. $\iint_R \ y\sin(xy)dxdy$ where $R = [1,2] \times [0,\pi]$ equals
    (a) π (b) 2π (c) 0 (d) 1
13. If $f: [0,1] \to \mathbb{R}$ is continuous then $\iint_S \ f(y)e^x dxdy, \ S = [0,1] \times [0,1] \ equals$
    (a) $(e-1) \ \int_0^1 f(y)dy$ (b) $e\int_0^1 f(y)dy$ (c) $\left(\frac{e^2}{2} - e\right)\int_0^1 f(y)dy$ (d) None of these.

14. $\iint_S \; e^{x/y} dxdy \; where \; S = \{(x,y) \in \mathbb{R}^2 : 1 \le y \le 2, y \le x \le y^3\}$ equals

(a) $\frac{e^2}{2} - \frac{e}{2}$ (b) $\frac{e^4}{2} - \frac{e^2}{2}$ (c) $\frac{e^2-1}{2}$ (d) None of these.

15. $\iint_S \; e^{\sin x \cos y} dxdy \; where \; S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 4\}$ lies between

(a) $4\pi e^2$ and $4\pi e^3$ (b) e^π and $e^{2\pi}$ (c) $\frac{4\pi}{e}$ and $4\pi e$ (d) None of these.

16. f is continuous on [0,1] and $\int_0^1 f(x)dx = 0$, then $\int_0^1 \int_0^x f(x)f(y)dydx$

(a) Depends on $f(y)$ (b) $\frac{1}{2}$ (c) 0 (d) cannot be evaluated.

17. $\iint_S \; (x - 3y^2)dxdy$ where $S = [0,2] \times [1,2]$ equals

(a) 12 (b) -12 (c) 6 (d) 0

18. Let $A(x) = \int_0^2 f(x,y)dy$ and $B(y) = \int_0^1 f(x,y)dx$ where $f(x,y) = x^2y^3$, then

(a) $A(x) = 3x^2, B(y) = y^4/4$ (b) $A(x) = x^4, B(y) = y^3$

(c) $A(x) = 4x^2, B(y) = y^3/3$ (d) None of the above.

19. The value of the integral $\iint_R \sqrt{x^2+y^2} \; dx \, dy$ where R= $\{(x,y) \in \mathbb{R}^2 : x \le x^2 + y^2 \le 2x\}$is

(a) 0 (b) 7/9 (c) 14/9 (d) 28/9

20. If $R = [0,1] \times [0,1]$, then $\iint_R \; e^{-x^2-y^2} dxdy$ lies between

(a) -1 and 0 (b) 0 and $\frac{1}{e^2}$ (c) $1/e$ and 1 (d) None of these.

21. f is continuous on $[0,1]$ and $\int_0^1 \; f(x)dx = 0$, then $\int_0^1 \int_0^x \; f(x)f(y)dydx$ is

(a) depends on $f(y)$ (b) $\frac{1}{2}$ (c) 0 (d) cannot be evaluated

22. $f(x,y) = \begin{cases} 2 & 1 \le x < 3 \quad 0 \le y \le 2 \\ 3 & 3 \le x \le 4 \quad 0 \le y \le 2 \end{cases}$ then,

(a) f is not integrable on $[1,4] \times [0,2]$

(b) $\int_0^2 \int_1^4 \; f = 5$.

(c) $\int_0^2 \int_1^4 \; f = 14$.

(d) None of these.

23. Let $f(x,y) = \sin\left(\frac{1}{x+y}\right), g(x,y) = \frac{1}{x+y}$ and $D = \{(x,y): x^2 + y^2 \le 1\}$.

Then which of the following statement is true.

(a) f and g are Riemann integrable over D.

(b) f is Riemann integrable over D, but g is not Riemann integrable over D.

(c) g is Riemann integrable over D, but f is not Riemann integrable over D.

(d) Both f and g are not Riemann integrable over D.

24. $f(x,y) = \begin{cases} 0 \;\; if \; x,y \; \in \mathbb{Q} \cap R \\ 3 \;\; if \; otherwise \end{cases}$ where $R = [0,1] \times [0,1]$. Then

(a) f is continuous at $(0,0)$ (b) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

(c) f is integrable over R (d) f is not integrable over R

25. If $f(a) = \int_a^{a^2} \frac{\sin ax}{x} \; dx$ then $f'(a)$ is

(a) $\int_a^{a^2} \frac{a \cos ax}{x} \; dx$ (b) $\int_a^{a^2} \cos ax \; dx$

(c) $\int_a^{a^2} \cos ax \; dx + 2 \sin a^3 - \frac{\sin a^2}{a}$. (d) None of the above

26. If $g(x) = \int_0^1 \frac{\sin xy}{y}\, dy$ on any interval $[a, b]$ not containing zero then $g'(x)$ equals

(a) $\frac{\cos x}{x}$ (b) $\frac{\sin x}{x}$ (c) $\frac{\cos y}{y}$ (d) None of the above.

27. $f(x,y) = \begin{cases} \frac{sinx}{x} & x \neq 0 \\ 0 & otherwise \end{cases}$

(a) $\iint_R f = 1$
(b) $\iint_R f = \cos 1 - 1$
(c) $\iint_R f = 1 - \cos 1$
(d) None of these.

28. The triple integral $\int\int\int_V dV$ where V is he region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$ can be expressed as an iterated integral in the order $dydzdx$ as

(a) $2\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+z^2}^4 dydzdx$ (b) $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^4 dydzdx$

(c) $2\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+z^2}^4 dzdydx$ (d) None of these.

29. The triple integral $\int_0^1 \int_0^x \int_0^y xy^2z^3dxdydz$

(a) $\frac{1}{90}$ (b) $\frac{1}{50}$ (c) $\frac{1}{45}$ (d) $\frac{1}{10}$

30. The value of $\int_0^1 (x+t)^2\, dx$ is

(a) $\frac{(t+1)^3}{3}$ (b) $t^2 + t - 1/3$ (c) $t^2 - 2t - 1/3$ (d) None of these.

31. The value of $\int_0^1 \log(xt)dx$ *is*

(a) $\log(1+t)$ (b) $2\log t$ (c) $\log t$ (d) None of these

32. If $g(x) = \int_0^1 \log(x^2 + y^2)dy \quad x \neq 0$. then $g'(x)$ equals

(a) 0 (b) 1 (c) $2tan^{-1}\frac{1}{x}$ (d) does not exist.

Descriptive Questions

(I) Use Fubinis theorem to evaluate $\iint_S f$. Sketch the region S of integration. Write both the iterated integrals.

1. $f(x,y) = x^2y$ and S is bounded by the lines $x = 2, x = 4, x = 2y$ and $x = y^2$.
2. $f(x,y) = x + y$ and S is defined by the parabola $y = x^2$ and $y = 1 - x^2$.
3. $f(x,y) = x + y$ and S is bounded by the lines $x + y = 1, x + y = 3$ and co-ordinate axes.
4. $f(x,y) = x + y$ and S is defined by $S = \{(x,y): |x| \leq 1, 0 \leq y \leq 1 + |x|\}$.
5. $f(x,y) = (1+x)\sin y$ and S is the trapezoid with vertices $(0,0), (1,0), (1,2)$ and $(0,1)$.
6. $f(x,y) = \frac{2x}{1+x^2+y^2}$ and S is the region in the first quadrant defined by $x^2 = 2y$ and $x = 2$.
7. $f(x,y) = x\sin y$ and S is given by $S = \{(x,y): 0 \leq x \leq \cos y, 0 \leq y \leq \pi\}$.
8. $f(x,y) = x^3y^2$ and S is the disk $x^2 + y^2 \leq a^2$.
9. $f(x,y) = xy^2$ and S is the region above the lines $y = 1 - x$ and inside the circle $x^2 + y^2 = 1$.
10. $f(x,y) = x^2 - xy$ and S is the region enclosed by $y = x$ and $y = 3x - x^2$.

11. $f(x,y) = x - y$ and S is the region above X-axis bounded by $y^2 = 3x$ and $y^2 = 4 - x$.
12. $f(x,y) = \frac{1}{\sqrt{2y-y^2}}$ is the region in the first quadrant bounded by $x^2 = 4 - 2y$.

(II) Evaluate the following integrals by reversing the order of integration. Sketch the region of integration.

1. $\int_0^9 \int_{\sqrt{y}}^3 \sin \pi x^3 dxdy$
2. $\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2} dxdy$
3. $\int_0^2 \int_{1+y^2}^5 ye^{(x-1)^2} dxdy$
4. $\int_0^2 \int_0^{\sqrt{\frac{4-x^2}{2}}} (x^2 + 4y^2) dydx$
5. $\int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2 - y^2)^{3/2} dydx$
6. $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{1 + y^3} dydx$
7. $\int_1^{\sqrt{2}} \int_{x^2}^2 \frac{1}{y^2} e^{x/\sqrt{y}} dydx$
8. $\int_0^1 \int_{\sqrt{x}}^3 e^{y^3} dydx$
9. $\int_1^{\sqrt{3}} \int_x^{\sqrt{3}} \frac{x}{(x^2+y^2)^{3/2}} dydx$
10. $\int_1^9 \int_{\sqrt{y}}^3 \frac{e^{(x^2-2x)}}{x+1} dxdy$
11. $\int_0^3 \int_1^{\sqrt{4-y}} (x + y) dx\, dy$
12. $\int_0^4 \int_{\sqrt{4-x}}^2 e^{y^3} dydx$

(III) Using double integration, find the area of the region S in $\mathbb{R}^2$, in the following examples:

1. S is bounded by the parabola $y = x^2$ and line $y = 2x + 3$.
2. S is bounded by the parabola $y = 9 - x^2$ and $y = x^2 + 1$.
3. S is bounded by the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.
4. S is the interior of the quadrilateral with the vertices $(1,0), (4,1), (3,3)$ and $(2,2)$.
5. S is bounded by the parabolas $y = x^2$ and $y = 4x - x^2$.
6. S is bounded by $y = \sin x$ and $y = \cos x$ for $0 \le x \le \frac{\pi}{4}$.

IV. Evaluate the following Triple integrals

1. $\int_0^1 \int_0^z \int_0^{x+z} 6xzdydxdz$
2. $\int_1^3 \int_x^{x^2} \int_0^{logz} xe^y dydx$
3. $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy$
4. $\int_1^e \int_1^e \int_1^e \frac{1}{xyz} dxdydz$
5. $\int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dzdydx$
6. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{y+1} dzdydx$

(IV) Evaluate the following triple integrals.

1. $\iiint_S zx \sin xy dV$ where S is the parallelepiped between the graphs of $z = 0$ and $z = 2$ on the rectangular region R in the XY-plane bounded by the lines $x = \frac{1}{6}, x = 1, y = 0$ and $y = \pi$.
2. $\iiint_S xyz dV$ where S is the bounded by the three co-ordinate planes and the plane $x + y + z = 1$.
3. $\iiint_S (x + 1) dV$ where S is the solid region between the graphs of the surface $z = -y^2$ and $z = x^2$ on the region R in the XY-plane bounded by $y = 0$ and $y = x$ for $0 \le x \le 1$.
4. $\iiint_S xy \sin yx dV$ where S is the rectangular box defined by the inequalities $0 \le x \le \pi, 0 \le y \le 1, 0 \le z \le \frac{\pi}{6}$.
5. $\iiint_S xyz dV$ where S is the solid in the first octant bounded by the parabolic cylinder $z = 2 - x^2$ and the planes $z = 0, y = 0, y = x$.
6. $\iiint_S zy dV$ where S is the solid bounded above by the plane $z = 1$ and below by the cone $z = \sqrt{x^2 + y^2}$.
7. $\iiint_S z dV$ where S is the solid region bounded above by the sphere $x^2 + y^2 + z^2 = 9$, below by the plane $z = 0$ and on the sides by the plane $x = -1, x = 1, y = -1$ and $y = 1$.
8. $\iiint_S y dV$ where S is the solid enclosed by the planes $z = 0, z = y$ and the parabolic cylinder $y = 1 - x^2$.
9. $\iiint_S y dV$ where S is the solid defined by the inequalities $\frac{\pi}{6} \le y \le \frac{\pi}{2}, y \le x \le \frac{\pi}{2}, 0 \le z \le xy$.
10. $\iiint_S xyz dV$ where S is the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying in the first octant.

Paper - I Change of Variables in Double and Triple Integrals Practical -2

Objective Questions

1. D is the closed region in the XY plane bounded by $y=\sqrt{1-x^2}$ and the x-axis.If R is the region in the $r-\theta$ plane whose image is D under the transformation $x=r\cos\theta,\ y=r\sin\theta$ then R is
   (a) $\{(r,\theta)/0<r<\sqrt{2}, 0\le\theta\le 2\pi\}$ (b) $\{(r,\theta)/0<r<1, 0\le\theta\le 2\pi\}$
   (c) $\{(r,\theta)/0<r<1, 0\le\theta\le \pi/2\}$ (d) $\{(r,\theta)/0<r<1, 0\le\theta\le \pi\}$

2. The double integral $\int\int_S f(x,y)\,dxdy$ where $S=\{(x,y)/x^2+y^2\le 2x\}$, expressed as an iterated integral in polar coordinates is
   (a) $\int_0^{2\pi}\int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta)r\ drd\theta$ (b) $\int_0^{2\pi}\int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta)\ drd\theta$
   (c) $\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta)r\ drd\theta$ (d) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta)r\ drd\theta$

3. $S=\{(x,y)/a^2\le x^2+y^2\le b^2\}$ with $0<a<b$. Then $\int\int_S f(x,y)\,dxdy$ expressed in polar coordinates is
   (a) $\int_0^{2\pi}\int_b^{a} f(r\cos\theta, r\sin\theta)r\ drd\theta$ (b) $\int_0^{2\pi}\int_a^{b} f(r\cos\theta, r\sin\theta)r\ drd\theta$
   (c) $2\int_a^{b}\int_0^{\pi} f(r\cos\theta, r\sin\theta)r\ d\theta dr$ (d) None of these.

4. The integral $\int_0^2\int_x^{x\sqrt{3}} f\left(\sqrt{x^2+y^2}\right)\ dydx$ in polar coordinates is
   (a) $\int_0^{\pi/4}\int_0^{2\sec\theta} f(r)r\ drd\theta$ (b) $\int_0^{\pi/3}\int_0^{2\sec\theta} f(r)r\ drd\theta$
   (c) $\int_{\pi/4}^{\pi/3}\int_0^{2\sec\theta} f(r)r\ drd\theta$ (d) $\int_{\pi/4}^{\pi/3}\int_0^{2\cos\theta} f(r)r\ drd\theta$

5. The area of the ellipse $4x^2+9y^2=36$ is
   (a) $\int_0^{\pi/2}\int_0^{6/\sqrt{4+5sin^2\theta}} rdrd\theta$ (b) $2\int_0^{\pi/2}\int_0^{6/\sqrt{4+5sin^2\theta}} rdrd\theta$
   (c) $4\int_0^{\pi/2}\int_0^{6/\sqrt{4+5sin^2\theta}} rdrd\theta$ (d) None of these.

6. The volume of the region bounded by $z=x^2+y^2,\ z=0, x=-a, y=a$ and $y=-a$ is
   (a) $\frac{4a^4}{3}$ (b) $\frac{8a^4}{3}$ (c) $\frac{16a^4}{3}$ (d) None of these.

7. The volume V of the solid above the region $R=\{(r,\theta)/1\le r\le 3,\ 0\le\theta\le\pi/4\}$ and under the surface $z=e^{x^2+y^2}$ is
   (a) πe (b) $\pi e(e-1)$ (c) $\frac{\pi}{8}(e^9-e)$ (d) $\frac{\pi}{8}e$.

8. If D is a plate defined by $1\le x\le 2, 0\le y\le 1$ and the density is ye^{xy}, then mass of the plate is
   (a) e (b) $\frac{e^2}{2}$ (c) $\frac{e^2}{2}-e$ (d) $\frac{e^2}{2}-e+\frac{1}{2}$

9. The centroid of the region bounded above by the line $y=1$ and bounded below by the curve $y=x^2/4$ is
   (a) $(0,\ 3/5)$ (b) $(1, 3/5)$ (c) $(2, 3/5)$ (d) $(-1, 3/5)$

10. The centroid of the rectangle bounded by the co-ordinate axes and the lines $x=a$ and $y=b$ has its centroid at
    (a) $(a/4,\ b/4)$ (b) $(a/2, b/2)$ (c) $(a/2, b/4)$ (d) None of these.

11. The moment of inertia of a homogeneous disk D, center at origin and radius a with density ρ about the origin is $\frac{\pi\rho a^4}{2}$. Then the moment of inertia of this disk about y-axis is
    (a) 0 (b) $\frac{\pi\rho a^4}{2}$ (c) $\frac{\pi\rho a^4}{4}$ (d) $\frac{\pi\rho a^4}{8}$

12. The density ρ of a region D is given by $\rho(x,y) = k, k$ constant. Then the center of mass D

(a) depends on ρ for some value of k (b) depends on ρ for any value of k

(c) does not depends on ρ (d) is located at (0, 0)

13. The volume of the solid bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2, y = 2$ and the three co-ordinate planes is given by the expression

(a) $\int_{-4}^{4}\int_{-4}^{4}(16 - x^2 - 2y^2)\,dxdy$ (b) $\int_{-2}^{2}\int_{-2}^{2}(16 - x^2 - 2y^2)\,dxdy$

(c) $\int_{0}^{2}\int_{0}^{2}(16 - x^2 - 2y^2)\,dxdy$ (d) $\int_{0}^{4}\int_{0}^{4}(16 - x^2 - 2y^2)\,dxdy$

14. The iterated triple integral $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2 + y^2)\,dzdydx$ in cylindrical coordinates is

(a) $\int_{0}^{2\pi}\int_{0}^{2}\int_{0}^{r} r^3\,dzdrd\theta$ (b) $\int_{0}^{2\pi}\int_{0}^{2}\int_{r}^{2} r^2\,dzdrd\theta$

(c) $\int_{0}^{2\pi}\int_{0}^{2}\int_{r}^{2} r^3\,dzdrd\theta$ (d) $\int_{0}^{2\pi}\int_{0}^{2}\int_{r}^{2} r\,dzdrd\theta$

15. A region R bounded by the coordinate axes and $x + y = 1$ in the first quadrant is the image of a region S lying in the uv plane under the transformation $u = x + y,\ v = x - y$. Then the area of the region S is

(a) 1 (b) 1/2 (c) $\sqrt{2}$ (d) Data insufficient

16. S is the region in the first quadrant bounded by the curve $xy = 1,\ xy = 2, y = x, y = 4x$.
If $u = xy, v = y/x$. Then $\iint_S f(x, y)\,dxdy$ becomes

(a) $\int_1^2\int_1^4 \frac{f(u)}{v}\,dvdu$ (b) $\int_1^2\int_1^4 \frac{f(u)}{2v}\,dvdu($

(c) $\log 2 \int_1^2 f(v)\,dv.$ (d) $\log 2 \int_1^2 f(u)\,du.$

17. $S = \{(x, y)/|x| + |y| \leq 1\}$. If $u = x + y, v = -x + y, then \iint_S f(x + y)dxdy$ equals.

(a) $\int_{-1}^{1}\int_{-1}^{1} f(u)dvdu$ (b) $\int_{-1}^{1}\int_{-1}^{1} \frac{f(u)}{4}dvdu$

(c) $4\int_{0}^{1}\int_{0}^{1} f(u)dvdu$ (d) $\int_{-1}^{1} f(u)du$

18. The expression for mass of a solid inside the cylinder $x^2 + y^2 = a^2$ and between the planes $z = 0$ and $z = h$ in the first octant with density x is

(a) $\int_0^h\int_0^a\int_0^{\sqrt{a^2-y^2}} y\,dxdydz$ (b) $\int_0^h\int_0^a\int_0^{\sqrt{a^2-y^2}} x\,dxdydz$

(c) $\int_0^h\int_0^a\int_0^{\sqrt{a^2-y^2}} x^2\,dxdydz$ (d) $\int_0^h\int_0^a\int_0^{\sqrt{a^2-y^2}} y^2\,dxdydz$

19. The expression for moment of inertia about the z-axis of homogeneous tetrahedron bounded by the planes $z = x + y, x = 0, y = 0, z = 1$ with volume density μ is

(a) $\mu\int_0^1\int_0^{1-y}\int_{x+y}^{1}(x^2 + y^2)\,dzdxdy$ (b) $\mu\int_0^1\int_0^{1-y}\int_{x+y}^{1} dzdxdy$

(c) $\mu\int_0^1\int_0^{1-y}\int_{x+y}^{1} z^2\,dzdxdy$ (d) None of the above.

20. The integral expression for each mass of the solid in the first octant bounded by the cylinder $x^2 + y^2 = 1$ and the plane $y = z, x - 0$ and $z = 0$ with density $\rho(x,y,z) = 1 + x + y + z$ is

(a) $\int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{x}(1 + x + y + z)\,dzdxdy$ (b) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{y}(1 + x + y + z)\,dzdxdy$

(c) $\int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{1}(1 + x + y + z)\,dzdxdy$ (d) None of the above.

21. The moment of inertia relative to the xz plane of a three dimensional region D with density ρ at each point is

(a) $\iiint_D x^2\rho\,dV$ (b) $\iiint_D x\rho\,dV$ (c) $\iiint_D y^2\rho\,dV$ (d) $\iiint_D y\rho\,dV$

22. The moment of inertia relative to the z-axis of a three dimensional region D with constant density 1 in spherical co-ordinates is

(a) $\iiint_D \rho^2 \sin^3\phi \, d\rho d\phi d\theta$ (b) $\iiint_D \rho^4 \sin^3\phi \, d\rho d\phi d\theta$

(c) $\iiint_D \rho^3 \sin^3\phi \, d\rho d\phi d\theta$ (d) $\iiint_D \rho^4 \sin^2\phi \, d\rho d\phi d\theta$

23. The moment of inertia of a three dimensional region D with constant density 1 in cylindrical co-ordinates is

(a) $\iiint_D z \, dzdrd\theta$ (b) $\iiint_D rz \, dzdrd\theta$ (c) $\iiint_D rz^2 \, dzdrd\theta$ (d) $\iiint_D r \, drdzd\theta$

24. 25. If D is the sphere $x^2+y^2+z^2 \le 9$ then $\iiint_D dV$ is equal to

(a) $6^3\pi$ (b) 18π (c) $6^2\pi$ (d) $6^4\pi$

25. If D is the unit sphere $x^2+y^2+z^2 \le 1$ then $\iiint_D z \, dV$ is equal to

(a) 0 (b) $\frac{2}{3}\pi$ (c) $\frac{4}{3}\pi$ (d) None of these

26. The volume of the portion of the solid cylinder $x^2+y^2 \le 2$ bounded above by the surface $z = x^2+y^2$ and below by the xy plane is

(a) π (b) 2π (c) 8π (d) 4π

Descriptive Questions

I. Evaluate the iterated integrals by converting to polar co-ordinates.

1. $\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{dxdy}{1+x^2+y^2}$
2. $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{x^2+y^2} \, dydx$
3. $\int_0^1 \int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} (x^2+y^2) \, dydx$
4. $\int_0^1 \int_{\sqrt{3}y}^{\sqrt{4-y^2}} \sqrt{x^2+y^2} \, dxdy.$
5. $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} (x^2+y^2) \, dydx$
6. $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dxdy$
7. □ $\int_{-1}^0 \int_{-1-x^2}^{\sqrt{1-x^2}} x \, dydx$
8. $\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} xy \, dxdy$
9. $\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2+y^2) \, dxdy$
10. $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dydx$

II. Evaluate the double integral as an iterated integral in polar co-ordinates.

1. $\iint_S (x+y)dA$ where S is the region in the first quadrant bounded by the lines $y=0, x=\frac{y}{\sqrt{3}}$ and the circle $r=2$.
2. $\iint_S \frac{dA}{1+x^2+y^2}$ where S is the sector in the first quadrant bounded by $y=0, y=x,$ and $x^2+y^2=4$
3. $\iint_S 3ydA$ where S is the region in the first quadrant bounded by the circle $(x-1)^2+y^2=1$ and below by the line $y=x$.

4. $\iint_S dA$ where S is the region in the first quadrant of the circle $x^2 + y^2 - 8y = 0$ cut by the line $y = \sqrt{3}x$.

5. $\iint_S dA$ where S is the annulus between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$.

III. Using the cylindrical co-ordinates evaluate the following integrals:

1. $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{4-x^2-y^2} x^2 dzdydx$
2. $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{\sqrt{9-x^2-y^2}} x^2 dzdydx$
3. $\iiint_S \sqrt{x^2 + y^2 + z^2}\, dxdydz$ where S is the region bounded by the plane $z = 3$, and the cone $z = \sqrt{x^2 + y^2}$
4. $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{(x^2+y^2)^2}^{1} x^2 dzdydx$
5. $\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \int_{0}^{1} (x^2 + y^2)\, dzdxdy$
6. $\iiint_S (x^2 + y^2) dxdydz$ where S is the solid bounded by the surface $x^2 + y^2 = 2z$ and the plane $z = 2$.
7. $\iiint_S \sqrt{x^2 + y^2}\, d(x, y, z)$ where S is the solid region bounded by the cylinder $x^2 + y^2 = 4$ and the plane $z = 0, y + z = 4$.
8. $\iiint_S f(x, y, z) d(x, y, z)$ where $f(x, y, z) = \frac{xz}{1+x^2+y^2}$ and $S = \{(x, y, z) \in \mathbb{R}^3\ 1 \leq x^2 + y^2 \leq 3,\ 0 \leq z \leq 3\}$.
9. $\iiint_S f(x, y, z) d(x, y, z)$ where $f(x, y, z) = \frac{z}{1+x^2+y^2}$ and $S = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x^2 + y^2 \leq 3,\ x \geq 0, y \geq x, 1 \leq z \leq 5.\}$
10. $\iiint_S f(x, y, z) dzdydz$ *where* $f(x, y, z) = \frac{z}{1+x^2+y^2}$ and $S = \left\{(x, y, z) \in \mathbb{R}^3\ 1 \leq x^2 + y^2 \leq 3, x \leq \sqrt{3}\, y \leq 3x, x \geq 0, 0 \leq z \leq 3\right\}$.

IV. Using cylindrical co-ordinates find the volume of the solid region S in $\mathbb{R}^3$ where

1. S is bounded by the paraboloid $x^2 + y^2 = 4 - z$ and the plane $z = 0$.
2. S is bounded above by the upper hemisphere of $x^2 + y^2 + z^2 = 25$, below by thw plane $z = 0$ and the laterally by the cylinder $x^2 + y^2 = 9$.
3. S is the bounded by the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = x^2 + y^2$.
4. S is the solid bounded above by the paraboloid $z = 5 - x^2 - y^2$ and below by the paraboloid $z = 4x^2 + 4y^2$.
5. S is the solid bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 2y$.

V. Evaluate the following integrals using spherical co-ordinates:

1. $\iiint_S e^{(x^2+y^2+z^2)^{3/2}}\, dxdydz$ where S is the unit sphere centered at origin.
2. $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2 + y^2 + z^2}\, dzdydx$.

3. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{1+x^2+y^2+z^2}\, dzdydx.$

4. $\iiint_S \frac{dxdydz}{(x^2+y^2+z^2)^{3/2}}$ where S is bounded by the sphere $\rho = a$ and $\rho = b; (a > b > 0)$.

5. $\iiint_S x^2 + y^2 dxdydz$ where S is the portion of the sphere $x^2 + y^2 + z^2 = a^2$ cut by the cone $x^2 + y^2 = 3z^2$.

6. $\iiint_S e^{-\left(\frac{x^2+y^2+z^2}{a^2}\right)^{3/2}} dxdydz$ where S is the spherical region given by $x^2 + y^2 + z^2 < a^2$.

7. $\iiint_S xyz\, dx\, dy\, dz$ where S is the solid in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 4$.

8. $\iiint_S x^2 dV$ where S is the solid region bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the upper half of the cone $z^2 = x^2 + y^2$.

VI. Using Spherical / Cylindrical Coordinates find the volume of the solid region S in $\mathbb{R}^3$ in the following examples:

1. S is the sphere $x^2 + y^2 + z^2 = a^2$
2. S is the region between two concentric spheres $\rho = a$ and $\rho = b; (b > a > 0)$.
3. S is above the cone $z^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 2az, (a > 0)$.
4. S is the volume cut from the sphere $\rho = a$ by the planes $\theta = 0$ and $\theta = \frac{\pi}{6}$ in the first octant.
5. S is the volume of the solid region bounded by the surface $\rho = \cos\phi$
6. S is the solid within the sphere $x^2 + y^2 + z^2 = 9$ outside the cone $z = \sqrt{x^2 + y^2}$ and above the XY-plane.
7. S is bounded by the cylinder $x^2 + y^2 = 25$ and the planes $z = 0, z = 8 - x - y$.
8. S is the solid enclosed between the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = y + 1$.
9. S is enclosed between the cylinder $x^2 + y^2 = 9$ and the plane $z = 1$ and $x + z = 5$.
10. S is enclosed by the paraboloids $z = 5x^2 + 5y^2$ and $z = 6 - x^2 - y^2$
11. S is enclosed by the surface $z = x^2 + y^2$ above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 2y$.

VII. Evaluate the following integrals by a suitable change of variables.

1. $\iint_S x^2 y dA$ where S is the region bounded by the lines $2x - y = 1, 2x - y = -2, x + 3y = 0, x + 3y = 1$.
2. $\iint_S \left(\frac{x-2y}{x+2y}\right)^3 dA$ where S is the region bounded by the lines $x - 2y = 1, x - 2y = 2, x + 2y = 1, x + 2y = 3$.
3. $\iint_S (x^2 + y^2)\, dxdy$ where S is the region in the XY-plane bounded by the curves $x^2 - y^2 = 1,\ x^2 - y^2 = 2, xy = 2, xy = 4$.
4. $\iint_S dxdy$ where S is the region bounded by the curves $xy = 4, xy = 8, xy^3 = 5, xy^3 = 15$.
5. $\iint_S (x + y) dxdy$ where S is the region in the XY-plane bounded by the curves $x^2 - y^2 = 1, x^2 - y^2 = 2,$ and the lines $y = x - 3, y = x - 1$.

6. $\iint_S \frac{\sin(x-y)}{\cos(x+y)}\, dxdy$ where S is the triangular region bounded by the parallelogram with vertices $(\pi,0),(2\pi,\pi),(\pi,2\pi)$ and $(0,\pi)$.

7. $\iint_R (y-x)\, dA$ *where* R is the region bounded by the straight lines $2x+3y-1=0, 2x+3y-3=0, x+2y=0, x+2y-5=0$.

8. $\iint_R \cos\left(\frac{x-y}{x+y}\right) dA$ *where* R is the region bounded by the coordinate axes and line $x+y=1$.

9. $\iint_R (x-y)e^{x^2-y^2}\, dA$ Where R is the rectangular region enclosed by the lines $x-2y=1, x-2y=4, 2x+2y=1, 2x+2y=3$.

10. $\iint_R \frac{y-4x}{y+4x}\, dA$ where R is the region bounded by the lines $y=4x, y=4x+2, y=2-4x, y=5-4x$.

VIII. Evaluate the following integrals using the indicated transformations:

1. $\iint_S (x^2+y^2)\, dxdy$ where S is the region bounded by the ellipse $4x^2+y^2=4$. Take $u=x$ and $v=\frac{y}{2}$.
2. $\iint_S xy^2 dA$ where S is the region bounded by the lines $x-y=2, x-y=-1, 2x+3y=1$ and $2x+3y=0$. Take $x=\frac{1}{5}(3u+v)$ and $y=\frac{1}{5}(u-2v)$.
3. $\iint_S e^{y-x}\, dA$ where S is the region bounded by the lines $2y=3x, y=2x$, and $y=x+1$. Take $x=u+v$ and $y=u+2v$.
4. $\iint_S \frac{y}{x}e^{x^2-y^2}\, dA$ where S is the region in the first quadrant bounded by the hyperbolas $x^2-y^2=1, x^2-y^2=4$, and the lines $x=2y, x=\sqrt{2}y$. Take $x=u\sec v$ *and* $y=u\tan v$ for $u>0$ and $0<v<\frac{\pi}{2}$.
5. $\iint_R e^{xy}\, dA$ where R is the region bounded by lines $2y=x,\ y=x$ and hyperbolas $xy=1$ and $xy=2$. Take $u=\frac{y}{x}\ \ v=xy$.

Paper - I **Line Integrals of Scalar and Vector Fields** **Practical -3**

Objective Questions

1. $f : IR \to IR^2, (f(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$. The image of [0, 1] is
   a) One full circle. b) an arc of a circle.
   c) an arc of a parabola d) none of the these

2. $f : IR \to IR^2, (f(t) = (e^t + e^{-t}, e^t - e^{-t})$. The image of [0, 1] is an
   a) an arc of a circle. b) an arc of a parabola
   c) An arc of a hyperbola d) none of these.

3. $I = \int \bar{F}\, dr$ $were$ $\bar{F} = (xy, yz, zx)$ $from$ $(0,0,0)$ to $(1,1,1)$. Then I is
   a) 0. b)1. c) $1/2$. d) none of these.

4. The value of the line integral $\int_C (x^2 + y^2)\, d\bar{r}$ where C is the arc $x^2 + y^2 = 1$ from $(0, 1)$ to $(1, 0)$ in clockwise direction is
   a) $\pi/2$. b) $-\pi/2$ c) π d) none of these.

5. The Cartesian representation of the curve having parametric equation $x = 3 + 5 \sin t, y = 1 + 2\cos t\,; 0 \le t \le 2\pi$ is
   a) $\frac{x^2}{25} + \frac{y^2}{4} = 1.$ b) $\frac{x^2}{9} + \frac{y^2}{1} = 1.$
   c) $\frac{(x-5)^2}{3} + \frac{(y-2)^2}{1} = 1.$ d) $\frac{(x-3)^2}{25} + \frac{(y-1)^2}{4} = 1.$

6. A parameterization α of a circle of radius 2 centered at the origin in the X Z plane is given by
   a) $\propto: [0, 2\pi] \to IR^3, \propto(t) = (2\, cost\, t, 2\sin t, 0)$
   b) $\propto: [0, 2\pi] \to IR^3, \propto(t) = (2\, cost\, t, 2\sin t, 1)$
   c) $\propto: [\pi, 3\pi] \to IR^3, \propto(t) = (2\, cost\, t, 0, 2\sin t)$
   d) $\propto: [0, 2\pi] \to IR^3, \propto(t) = (0, 2\, cost\, t, 2\sin t)$

7. The parametric equations $x = 2 + 3t^3$ $y = 4 + 7t^3$ elements.
   a) The curve $y = x^3, 0 \le x \le 1.$
   b) The curve $y^3 = x, 0 \le x \le 1.$
   c) The curve $x^3 - y^3 = 2, 0 \le x \le 1.$
   d) line having intercept on both the axes.

8. The parametric equations $x = \cos(\cos t), y = \sin(\cos t), t \in [0, \pi]$ describes.
   a) one full circle
   b) an arc of a circle in first quadrant
   c) one half circle above the XY-plane
   d) an arc of a circle in the first and fourth quadrant

9. The equation$x = \cos t, y = cost, 0 \le t \le \pi$ parameterizes
   a) an arc of a circle. b) an arc of a parabola
   c) a line segment d) a branch of a hyperbola.

10. $I = \int_C \frac{-y\,dx + x\,dy}{(x^2+y^2)^m}$ $where\ C:\ x^2+y^2 = r^2$. Then I is

a) 0. b) 2π. C) $\frac{2\pi}{r^{2m}}$ d) $\frac{2A}{r^{2m}}$; where A is area of the circle.

Descriptive Questions

I. Evaluate the integral of the scalar field f along the given path.

1. $f(x,y) = x^3 + y$, C is the curve $x(t) = 3t, y(t) = t^3; 0 \le t \le 1$.
2. $f(x,y,z) = x^2 + y^2 + z^2$; C is the curve given by $x(t) = \cos t, y(t) = \sin t, z(t) = t; 0 \le t \le \pi/2$.
3. $f(x,y,z) = x + y + z$ and $\gamma(t) = (\sin t, \cos t, t)$, $0 \le t \le 2\pi$
4. $f(x,y,z) = \cos z, \gamma(t) = (\sin t, \cos t, t), 0 \le t \le 2\pi$
5. $f(x,y,z) = x\cos z, \gamma(t) = ti + t^2 j, 0 \le t \le 1$
6. $f(x,y,z) = e^{\sqrt{z}}$, and $\gamma(t) = (1, 2, t^2), 0 \le t \le 1$
7. $f(x,y) = \sin x + \cos x$, C is the line segment from $(0,0)$ to $(\pi, 2\pi)$.
8. $f(x,y,z) = 2x + 9z$, C: $x(t) = t, y(t) = t^2, z(t) = t^3$; $0 \le t \le 1$.
9. $f(x,y,z) = x^2 + y^2 + z^2$
   C; $x(t) = \cos\ t, y(t) = \sin\ t, z(t) = t$; $0 \le t \le \pi/2$.
10. f(x, y, z) = x + y + z C is the line segment from (1, 2, 3) to (0, -1,1).

II. Evaluate the integral of the vector field F along the given path.

1. $F(x,y) = (x^2 - 2xy, y^2 - 2xy)$; C is the curve $y = x^2$ from (-1, 1) to (1, 1).
2. $F(x,y) = (y, x^2)$; C is the line segment from (0, -1) to (4, -1) and then to (4, 3).
3. $F(x,y,z) = (xz,\ y + z, x)$; $C: x(t) = e^t, y(t) = e^{-t}, z(t) = e^{2t}, 0 \le t \le 1$.
4. $F(x,y,z) = (2x + yz, xz, xy + 2z)$ C: $x^2 + y^2 = 1$; $z = 1$ from $(0,1,1)$ to $(1,0,1)$.
5. $F(x,y,z) = (x, z, -yx)$ C is the circular helix given by $x(t) = \cos\ t,\ y(t) = \sin t, z(t) = 2t$; $0 \le t \le 3\pi$.
6. $F(x,y,z) = (x,^2 - xy, 1)$; along each of the following curves:
   a) The straight line joining (0, 0, 0) to (1,1,1)
   b) The circle of radius 1, with centre at the origin and lying in the yz, plane, traversed counterclockwise as viewed from the positive x axis
   c) The parabola $z = x^2, y = 0$, between $(-1, 0, 1)$ and $(1, 0, 1)$
   d) The straight line between $(-1, 0, 1)$ and $(1, 0, 1)$

III. Evaluate the following line integrals

1. Evaluate the line integral $\int_C \sin z\, dx + \cos z\, dy - (xy)^{\frac{1}{3}}\, dz$ where C be parametrized by $x = cos^3\theta, y = sin^3\theta, z = \theta, 0 \le \theta \le 7\pi/2$.
2. Evaluate the line integral $\int_C x^2 dx + xy\, dy + dz,$ where C is parametrized by $c(t) = (t, t^2, 1), 0 \le t \le 1$.
3. Evaluate $\int_C F(r).dr$, where $F(x, y, z) = \sin zi + \cos\sqrt{yj} + x^3 k$ and C is the line segment from $(1, 0, 0)$ to $(0, 0, 3)$.
4. Evaluate $\int_C 2xydx + x^2 zdy + x^2 ydz$, where C is an oriented curve connecting $(1, 1,1)$ to $(1, 2,4)$.
5. Evaluate $\int_C (3x - y)\, dx + y\, dy$ where C is parametrised by $\propto: [\,0, \log 2] \rightarrow IR^2 \;\; \alpha\,(t) = (e^t - 1, e^t - 1)$
6. Evaluate the integral of the vector field $F\,(x, y) = (x^2 - 2xy, y^2 - 2xy)$; along the parabolic path C $y = x^2\ from\ (-1, 1)\ to\ (1, 1)$.
7. Evaluate the integral of the vector field $F\,(x, y, z) = \left(xz,\ \ y + z\,, x\right)$; along the given path $C\ is\ the\ curve\ x(t) = e^t, y\,(t) = e^{-t}, z(t) = e^{2t}; 0 \le t \le 1$.
8. Evaluate $\int_C \frac{1+y^2}{x^3}\, dx - \left(\frac{1+y^2}{x^3}\right) y\, dy$ where C is the straight line path joining $(1\,, 0)\ to\ (2, 0)\ and\ (2, 0)\ to\ (2, 1)$.

IV. Solve the following:

1. Calculate the work done by the force field $F(x, y, z) = xi + yj$ when a particle is moved along the path $(3t^2, t, 1), 0 \le t \le 1$.
2. Find the work done by force field when a particle is moved along the straight- line segment from $(0, 0, 1)$ to $(3, 1, 1)$.
3. Find the work which is done by the force field $F(x, y) = (x^2 + y^2)\,(i + j)$ around the loop $(x, y) = (\cos t, \sin t), 0 \le t \le 2\pi$.
4. Let $F = (z^3 + 2xy)i + x^2 j + 3xz^2 k$. Show that the integral of F around the square with vertices $(\pm 1, \pm 1, 0)$ is zero.

Paper - I **Green's Theorem, Conservative Field and Applications** **Practical 4**

Objective Questions

1) $F(x,y) = (x^2y^5, ax^by^c)$ is conservative in the plane then
   a) $a = \frac{1}{3},\ b = 1, c = 6$ b) $a = 5/3, b = 3, c = 4$.
   c) b & c exists but a does not exist. d) $a = 1, b = 2, c = 5$.

2) $F(x,y,z) = (2xy + y^2, x^2 + 2xy + z, y + e^{xz})$ then
   a) there exist a function $\phi(x,y,z)$ such that $F = \nabla\phi$
   b) there does not exist a function $\phi(x,y,z)$ such that $F = \nabla\phi$
   c) $\phi(x,y,z) = 2x^2y + 2xy^2 + 2y^2, F = \nabla\phi$
   d) $\phi(x,y,z) = x^2y + xy^2 + \frac{e^{xz}}{x} + yz, F = \nabla\phi$

3) The line integral $\int_c^{\bar{F}} \bar{F} \cdot d\bar{r}$; $\bar{F} = \frac{-y\,\hat{\imath} + x\hat{\jmath}}{x^2+y^2}$ and $C: x^2 + y^2 = a^2$.
   a) depends on a.
   b) does not exist as Green's Theorem is not applicable.
   c) is a constant independent of a.
   d) none of the above.

4) $I = \oint_c y\,dx + 2x\ dy$ where C is a closed curve of the region $x^2 + y^2 \leq a^2$ Then I is
   a) a^2 b) $\pi\ a^2$ c) 0. d) None of these.

5) $\oint_c P\,dx + Q\,dy = 0$ around every C is a closed path C in a simply continued region R then
   a) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ if P and Q are C^1 function.
   b) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ always.
   c) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.
   d) Nothing can be said about $\frac{\partial P}{\partial y}$ *and* $\frac{\partial Q}{\partial x}$.

6) $I = \oint_c (x+y)\hat{\imath} + (x-y)\hat{\jmath}$, Where C is the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ then I is
   a) $\pi\,ab$ b) 0. c) $\pi(a+b)$ d) ab.

7) $I = \int_c ydx + xdy$ where C is the path $(t^9, sin^9(\pi/2)); 0 \leq t \leq 1$ Then I is
   a) 1 b) 0. c) -1. d) π.

8) $\nabla f(x,y,z) = 2xyze^{x^2}\hat{\imath} + ze^{x^2}\hat{\jmath} + ye^{x^2}\hat{k}$ and $f(0,0,0) = 5$. Then $f(1,1,1)$.
   a) 5. b) e. c) $e + 5$. d) $3e$

9) $\int_c ydx + xdy$ along every closed curve C is
   a) 2π b) π c) $\pi/2$ d) None of these.

10) $P = \log(x^2+1) - 2xe^{-y}, Q = x^2e^{-y} - \log(y^2+1)$. Then

a) $\int_{C_1} Pdx + Qdy = \int_{C_2} Pdx + Qdy$ for any two curves C_1 & C_2 with same end points.

b) $\int_{C_1} Pdx + Qdy = \int_{C_2} Pdx + Qdy$ for any two curves $C1$ & $C2$ by Green's Theorem.

c) $\int_{C_1} Pdx + Qdy \neq \int_{C_2} Pdx + Qdy$ for any two curves C_1 & C_2.

d) None of these.

Descriptive Question

I. Find whether the following force field F is conservative .If so find $\emptyset$ so that $F = \nabla\emptyset$ and calculate the work done in the moving the particle from the point P to the point Q

1. $F(x,y) = (x^3y^4 + x, x^4y^3 + y)$; $P(1,1),\ Q(2,0)$.
2. $F(x,y) = (x^2 + y^2, 2xy)$; $P(1,0),\ Q(1,\ 1)$.
3. $F = (y\sin z,\ x\sin z, xy\cos z)$. $P(0,0,0),\ Q(\pi,\pi,\pi)$
4. $F = 2xi + 3yj + 4zk, P(1,-1,0),\ Q(2,0,1)$
5. . $F = (y+z)i + (x+z)j + (x+y)j, P(1,-1,0),\ Q(2,0,1)$

6. $F = e^{y+3z}(i + xj + 3xk), P(1,-1,0),\ Q(2,0,1)$.

II. Calculate the work done in the moving the particle from the point P to the point Q for the following force fields, showing first that they are conservative.

1.$F(x,y) = (x^2 + 4xy + 4y^2, 2x^2 + 8xy + 8y^2), P = (2,-1)$ and $Q = (-4,2)$.

2.$F(x,y) = \left(e^{y^2}\cos x, 2ye^{y^2}\sin x\right), P = \left(\frac{\pi}{2}, 0\right)$ and $Q = \left(\frac{\pi}{2}, 1\right)$.

3.$F(x,y,z) = (e^x\sin z + 2yz, 2xz + 2y, e^x\cos z + 2xy + 3z^2), P = \left(0, 1\ \frac{\pi}{2}\right)$ and $Q = \left(1, 0,\ \frac{\pi}{2}\right)$

4.$F(x,y,z) = (3x^2\sin xyz + x^3\ yz\cos xyz, x^4y\cos xyz)$; $P(1,-1,0)\ Q(2,0,1)$.

5.$F(x,y) = \left(y(e^{xy}+1), x(e^{xy}+1)\right); P(1,0)Q(1,1)$.

III. Consider the vector field $\vec{F}(x,y,z) = e^y\vec{\imath} + xe^y\vec{\jmath} + (z+1)e^z\vec{k}$.

1. Find a scalar function f such that $\vec{F} = \nabla f$.
2. Use part (1) to evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$. where C is the curve described by $\vec{r}(t) = t\vec{\imath} + t^2\vec{\jmath} + t^3\vec{k}$ for $0 \le t \le 1$.

IV. In the following problems show that the given line integral is independent of the path. Evaluate the line integral.

1. $\int_{(-1,2)}^{(3,1)} (y^2 + 2xy)dx + (x^2 + 2xy)\,dy$
2. $\int_{(0,0)}^{(1,\pi/2)} (e^x\ \sin\ y)\ dx + (e^x\ \cos\ y)\ dy$.

3. $\int_{(0,0,0)}^{(1,2,3)} 2x\,dx + (x^2 - z^2)dy - 2yz\,dz.$
4. $\int_{(1,0,0)}^{(0,1,1)} \sin y \cos x\,dx + \cos y \sin x\,dy + dz.$
5. $\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y\,dx + \left(\frac{1}{y} - 2x\sin y\right) dy + \frac{1}{z}\,dz$
6. $\int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+x^2}$

V. Verify Green's Theorem for the following examples:

1. $P(x,y) = 2x - y + 4, Q(x,y) = 5y + 3x - 6$ and C is the triangular with vertices $(0,0), (3,0)$ and $(3,2)$ having positively oriented boundary.
2. $F(x,y) = (2xy - x^2, x + y^2)$ and C is the region bounded by the closed curve Γ formed by $y = x^2$ and $y^2 = x$ in the anticlockwise direction.
3. $F(x,y) = (2x - y^3, -xy)$ and C is the positively oriented boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.
4. $F(x,y) = (-y, x)$ and C is the positively oriented boundary of the region defined by the lines $x = 0, y = 0, x + y = 1$ and $x + y = 2$.

VI. Use Greens theorm to evaluate the following line integrals:

1. $\oint 4x^2y\ dx + 2y\ dy$; C is the boundary of the triangle with vertices $(0,0)$, $(1,2)$ and $(0,2)$.
2. $\oint 2x\cos y\ dx + x^2 \sin y\,dy$; C is the boundary of the region R enclosed between $y = x^2$ *and* $y = x$.
3. $\oint -y\ dx + x\,dy$; C is the boundary of the region R enclosed between $x + y = 1, x + y = 2,\ x = 0$ *and* $y = 0$.
4. $\oint 4x^2\ y\,dx + 2y\,dy$; C is the boundary of the triangle with vertices $(0,0), (1,2)$ *and* $(0,2)$.
5. $\oint 2x\cos y\,dx + x^2 \sin y\,dy$; C is the boundary of the region R enclosed between $y = x^2$*and*$y = xx$.
6. $\oint (e^{3x} + 2y)\,dx + (x^2 + \sin y)\ dy$; where C is the rectangle with vertices $(2,1), (6,1), (6,4)$ and $(2,4)$.
7. $\oint (\cos x \sin y - xy)dx + \cos y \sin x\,dx$; where $C: x^2 + y^2 = 1$.
8. $\oint_C (y + 3x)dx + (2y - x)dy$ where C is the ellipse $x^2 + \frac{y^2}{4} = 1$.
9. $\oint_C (e^{-y} - 2x)dx + (xe^{-y} + \sin y)dy$ where is the first quadrant arc of the circle $x^2 + y^2 = \pi$.
10. $\oint_C F$ where $F(x,y) = (x^2 - 2xy, x^2y + 3)$ and C is the positively oriented boundary of the region, $y^2 = 8x$ and $x = 2$.

VII. Using Green's Theorem, find the area of the region D whose boundary is positively oriented simple closed curve, in the following examples:

1. D is the triangle with vertices $(1,1), (4,1)$ and $(4,9)$.
2. D lies in the first quadrant bounded by the lines $4y = x$ and $4y = y$ and the hyperbola $xy = 4$.
3. D is bounded by the lines $y = 1, y = 3, x = 0$ and the parabola $y^2 = x$.
4. $D = \left\{(x,y): \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} \leq 1, a \geq 0\right\}$
5. $D = \{(x,y): 2x^2 + 3y^2 \leq 1\}$

6. D is the interior of the circle C: $r(t) = (a\ \cos\ t)i + (a\ \sin\ t)j,\ 0 \le t \le 2\pi$

7.D is the interior of the ellipse C: $r(t) = (a\ \cos\ t)i + (b\ \sin t)j,\ 0 \le t \le 2\pi$

8.D is the asteroid $r(t) = (\cos^3 t)i + (\sin^3 t)j,\ 0 \le t \le 2\pi$

9.D is the region bounded by the closed curve $r(t) = t^2 i + ((t^2/3) - t)j, -\sqrt{3} \le t \le \sqrt{3}$

Paper - I **Evaluation of Surface Integrals** **Practical 5**

1) The equation $x = u^2 - v^2, y = 2uv, z = u^2 + v^2$ represents
a) a cone. b) a sphere. c) a circle. d) a Cylinder .

2) The equation $x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = 4 - r^2$ represents.
a) a cylinder. b) a sphere. c) a paraboloid. d) none of these.

3) For the cylinder $x = 3\cos t,\ y = y, z = 3\sin t$ at the point $(3/\sqrt{2}, 1, 3/\sqrt{2})$.
a) there is an unique unit normal vector.
b) there are two unit normal vectors.
c) there is no unit normal vector.
d) there are infinitely many unit normal vectors.

4) The equations $x = u + v, y = u - v, z = u^2 + v^2; 0 \le u \le 1, 0 \le v \le 1$.
a) a cone. b) a sphere. c) a paraboloid d) a Cylinder.

5) The equation $x = 5\cos\theta, y = 5\sin\theta, z = 7; 0 \le \theta \le 2\pi$ represents.
a) a straight line segment. b) a plane. c) a circle d) a Cylinder.

6) The surface integral of $F(x, y) = -yi + x\hat{j}$ on S where S is the disc in the XY plane with radius 2 oriented upwards and at the origin is
a) 1. b) –1. c) 0. d) None of these.

7) The surface area of the triangle with vertices (1, 0, 0), (0, 1, 0) and (0, 0, 1) is
a) $\sqrt{3}$. b) $\frac{\sqrt{3}}{2}$. c) $2\sqrt{3}$. d) $1/2$.

8) The surface integral of $F(x, y, z) = x^2\hat{\imath} + y^2\hat{\jmath} - z\hat{k}$ on the triangle with vertices (0, 0, 0), (0, 2, 0) and (0, 0, 3) is
a) 1. b) –1. c) 0. d) $1/2$

9) The surface area of the sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ is denoted by A. Then,
a) A depends on a, b, c and r. b) A depends only on a, b, c.
c) a depends only on r. d) None of these.

10) The equation $x = u + 2v, y = 2u - 3v, z = 3u + 4v$ describes.
a) a general plane. b) a plane passing through the origin.
c) a line in $\mathbb{R}^3$ d) none of these.

11) The magnitude of the fundamental vector product $\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}$ for surface $\bar{r}(u, v) = (u + v)\hat{\imath} + (u - v)\, i + 4k$ is
(a) $\sqrt{4 + v^2}$ (b) $\sqrt{4 + 128}v^2$ (c) $\sqrt{4v^2 + 1}$ (d) None of these.

12) The parametric representation of cylinder $x^2 + y^2 = 4,\ 0 \le z \le 1$ is given by
(a) $x = 2\cos u,\ \ y = 2\sin v,\ \ z = u^2 + v^2,\ \ 0 \le u \le 2\pi,\ \ 0 \le v \le \pi.$
(b) $x = 2\cos u,\ \ y = 2\sin u,\ \ z = u,\ \ 0 \le u \le 2\pi.$
(c) $x = 2\cos u,\ \ y = 2\sin u,\ \ z = z,\ \ 0 \le u \le 2\pi,\ \ 0 \le z \le 1.$
(d) None of the above.

13) The parameterization $x = \cosh u \cos v,\ \ y = \cosh u \sin v,\ \ z = \sin hu$, where $0 \le v \le 2\pi, -\infty < u < \infty$ represents

(a) an ellipsoid (b) a hyperboloid of one sheet

(c) a cylinder (d) None of these.

14) The fundamental vector product for the cone $x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = r,\ \ 0 \le \theta \le 2\pi,\ \ 0 \le r \le 1$ is

(a) $(-r\cos\theta,\ -r\sin\theta,\ r)$ (b) $(r\cos\theta,\ r\sin\theta,\ r)$

(c) $(-r\cos\theta,\ r\sin\theta,\ r)$ (d) $(r\cos\theta,\ -r\sin\theta,\ r)$

15) The area of surface of revolution of the curve $y = f(x)$ parameterized by $x = u, y = f(u)\cos v,\ \ z = f(u)\sin v,\ \ a \le u \le b,\ \ 0 \le v \le 2\pi$ is

(a) $\int\ |f(u)|\sqrt{1+(f'(u))^2}\ du$

(b) $2\pi\int\ |f(u)|\sqrt{1+(f'(u))^2}\ du$

(c) $\frac{1}{2\pi}\int_a^b f(u)\sqrt{1+(f'(u))^2}\ du$

(d) None of these.

16) $\iint_S\ xdS$ where S is the triangle with vertices (1, 0, 0), (0, 1, 0), (0, 0, 1) is

(a) $\sqrt{3}$ (b) $\frac{\sqrt{3}}{6}$ (c) $\frac{\sqrt{3}}{2}$ (d) None of these.

17) The flux of the vector field $\bar{r} = \widehat{x\imath} + \widehat{y\imath} + \widehat{zk}$ across the unit sphere $x^2 + y^2 + z^2 = 1$ equals

(a) $\frac{4}{3}\pi$ (b) $\frac{2}{3}\pi$ (d) $\frac{1}{3}\pi$ (d) None of these.

18) Let $\bar{F} = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$, where P, Q, R are continuously differentiable and S is the surface given by $z = g(x,y), (x,y)\ \in D$, then $\iint_S\ F.\ \hat{n}\,dS$ is given by

(a) $\iint_D\ \left(P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} + R\right)dxdy$ (b) $\iint_D\ \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)dxdy$

(c) $\iint_D\ \left(P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} - R\right)dxdy$ (d) None of these.

19) The centre of mass of a uniform hemispherical surface of radius a having parametric representation $\bar{r}(u,v) = a\cos u\cos v\,\hat{\imath} +\ a\sin u\cos v\,\hat{\jmath} +\ a\sin v\ \bar{k}$, $(u,v) \in [0, 2\pi] \times [0, \pi/2]$ is given by

(a) $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$ (b) $\left(0, 0, \frac{a}{2}\right)$ (c) $(0, 0, 0)$ (d) None of these.

20) The parameterized surface $\bar{r}\ (u,v)$ given by $x = x_0 + a_1u + b_1v,\ \ y = y_0 + a_2u + b_2v, z = z_0 + a_3u + b_3v$ represents (where $x_0, y_0, z_0, a_1, a_2, a_3, b_1, b_2, b_3$ are constants)

(a) A sphere with centre (x_0, y_0, z_0) (b) a cylinder

(c) an ellipsoid (d) a plane

DESCRIPTIVE QUESTIONS

(I) Evaluate surface integrals of scalar field f over S:

1. $f(x,y,z) = x+y+z$ and S is the cube centered at the origin $(-1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1)$
2. $f(x,y,z) = x^2$ where S is the part of the plane $x+2y+3z=6$ in the first octant.
3. $f(x,y,z) = x^2+y^2$ where S is the surface of the paraboloid $x^2+y^2 = 2-z$ above the XY-plane.
4. $f(x,y,z) = z$ and S is the upper hemisphere $x^2+y^2+z^2 = a^2$.
5. $f(x,y,z) = x+y+z$ and S is the portion of the plane $x+y=1$ in the first octant for which $0 \le z \le 1$
6. $f(x,y,z) = xz+2y$ on the surface whose parameterization is $r(u,v) = u^2\hat{\imath} + \frac{v^2}{2}\hat{\jmath} + uv\hat{k}$ where $0 \le u \le 1$ and $0 \le v \le 2$.
7. $f(x,y,z) = z^2$ where S is the portion of the cone $x^2+y^2 = z^2$ between the plane $z=1$ and $z=2$.
8. $f(x,y,z) = xyz$, S is the surface of the cone $z^2 = x^2+y^2$ between $z = 1$ and $z = 2$.
9. $f(x,y,z) = y^2$ and S is the cylinder $x^2+y^2 = 1, 0 \le z \le 1$ and its top and bottom.
10. $\iint_S xdS$, where S is the surface $y = x^2+4x, 0 \le x \le 2, 0 \le z \le 2$.

(II) Evaluate the surface integrals of vector field F over S:

1. $F(x,y,z) = (18z, -12, 3y)$ and S is the surface $2x+3y+6z = 12$ in the first octant.
2. $F(x,y,z) = (x,y,0)$ and S is the hemisphere above XY-plane
3. $F(x,y,z) = (x,y,z)$ and S is the piece of the cylinder with parameterization $r(x,y) = (\cos x, \sin x, y)$ where $(x,y) \in \left[0, \frac{\pi}{2}\right] \times [0,1]$.
4. $F(x,y,z) = y\hat{\imath} + 2\hat{\jmath} + xz\hat{k}$ and S is the parabolic cylinder $y = x^2$ bounded by $0 \le x \le 3, 0 \le z \le 2$.
5. $F(x,y,z) = (y, -x, xyz)$ and S is the surface bounding the region defined by $x^2+y^2 \le 2$ and $0 \le z \le x+2$ oriented outward.
6. $F(x,y,z) = y\bar{\imath} + x\bar{\jmath} + z^2\bar{k}$ S is helicoids with vector equation $\bar{r}(u,v) = u\cos v\,\bar{\imath} + u\sin v\,\bar{\jmath} + v\bar{k}$. $0 \le u \le 1, 0 \le v \le 2\pi$
7. $F(x,y,z) = (x,y,z)$; S is the paraboloid $z = x^2+y^2-1$; $-1 \le z \le 0$ oriented upwards.
8. $F(x,y,z) = (e^{-y}, -y, x\sin z)$ S is the surface parameterized by

   $\alpha(u,v) = (2\cos v, \sin v, u)$; $0 \le u \le 5, 0 \le v \le 2\pi$.
9. $F(x,y,z) = (0,y,1)$; S is the portion of the paraboloid $z = x^2+y^2$ below the plane $z=4$ oriented downwards.

10. $F(x,y,z) = (x,y,z)$; S is the surface parameterized by $\alpha\,(u,v) = (u\cos v, u\sin v, 1-u^2)$; $1 \le u \le 2, 0 \le v \le 2\pi$.

(III) Find the surface area of S, represented by the following surfaces:

1. S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
2. S is the surface of the paraboloid $z = 1 - x^2 - y^2$ cut by the plane $z = -3$.
3. S is the part of the surface $z = 9 - x^2 - y^2$ that lies above the ring $1 \le x^2 + y^2 \le 9$.
4. S is the partion of the upper hemisphere $x^2 + y^2 + z^2 = 2$ cut by the cylinder $x^2 + y^2 = 1$.
5. S is the area cut from the plane $x + y + z = 5$ by the cylinder whose walls are $x = y^2$ and $x = 2 - y^2$
6. S is parametrically given by $r(\theta, z) = (a\cos\theta, a\sin\theta, z)$ and $(\theta, z) \in [0, 2\pi] \times [-1, 1]$ with $a > 0$ is constant.
7. S is the torus parameterized by the equations $x = (a + \cos u)\cos v, y = (a + \cos u)\sin v, z = \sin u$ where $-\pi < u, v < \pi,\ a \ge 1$.
8. S is the surface of the cylinder $x^2 + y^2 = a^2$ which is cut out by the cylinder $x^2 + y^2 = a^2$
9. The part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.
10. The part of the plane $x + 2y + z = 4$ that lies inside the cylinder $x^2 + y^2 = 4$.
11. S is the portion cut from the paraboloid $y = 1 - x^2 - z^2$ by the plane $y = 0$.
12. S is the surface parameterized by $x = r\ \cos\theta, y = 2r\cos\theta, z = \theta, 0 \le r \le 1, 0 \le \theta \le 2\pi$.

Paper - I **Stoke's and Gauss divergence Theorem** **Practical 6**

Objective Questions

1. Let $\bar{F}(x,y,z) = \hat{y\imath} - \hat{x\jmath} + zx^3y^2\hat{k}$ and S denote the upper hemisphere $x^2+y^2+z^2=1, z>0$ with unit normal $\hat{n}$ having non- negative z –component. Then $\iint_S (\nabla \times \bar{F}).\ \hat{n}$ equals
   (a) 0 (b) -2π (c) 2π (d) 1

2. Let $\bar{F}(x,y,z) = ye^z\hat{\imath} + xe^z\hat{\jmath} + xye^z\hat{k}$ and S be the surface of unit sphere with outward normal $\hat{n}$. Then $\iint_S (\nabla \times \bar{F}).\hat{n}$ equals
   (a)4π (b) 12π (c) 16π (d) 0

3. Let $\bar{F}(x,y,z) = \hat{z\imath} - \hat{x\jmath} - \widehat{yk}$ and C be the triangle with vertices $(0,0,0), (0,2,0)$ and (0, 0, 2). Then $\int_C \bar{F}.d\bar{r}$ equals
   (a) 1 (b) -1 (c) -2 (d) 2

4. Let S denote an oriented smooth surface bounded by a closed curve C traversed counterclockwise. Let $\bar{r} = \hat{x\imath} + \hat{y\jmath} + \widehat{yk}$. If $\bar{A}$ is a constant vector and $\hat{n}$ be the unit outward normal to S. then $\oint_C (\bar{A} \times \bar{r}).dr$ equals
   (a)$\iint_S curl\,\bar{r}.\hat{n}dS$ (b) 0 (c) $2\iint_S \bar{A}.\hat{n}dS$ (d) None of these.

5. If S is a sphere and $\bar{F}$ is a vector field having continuous partial derivatives on an open region containing S, then $\iint_S curl\,\bar{F}.\hat{n}dS$ where $\hat{n}$ is unit outward normal
   (a) Depends on F (b) 4π (c) 2π (d) 0

6. If V is a simple solid region whose boundary surface is S and $\hat{n}$ is a unit outward normal to S. then for a harmonic function $\emptyset$ defined on a region containing S, $\iint_S D\emptyset\ \hat{n}dS$ equals
   (a) Volume of V (b) surface area of S (c) 0 (d) None of these.

7. If V is a simple solid region in $\mathbb{R}^3$ bounded by a smooth oriented surface S with $\hat{n}$ as outward unit normal. and $\bar{A}$ is a constant vector in $\mathbb{R}^3$, then $\int_S \bar{A}.\hat{n}dS$ equals
   (a) $||A||$ (b) (surface area of S) $||A||$ (c) (volume of V) $||A||$ (d) 0

8. Let $\hat{n}$ be a unit outward normal to a closed surface S which bounds a homogeneous solid V. then $\iint_S (x^2+y^2)(xi+yj).\hat{n}dS$ equals
   (a) $|V|$, the volume of V (b) $|S|$, the surface area of S
   (c) $4I_z$, where I_z denote the moment of inertia about z-axis (d) None of these.

9. The flux of $\bar{F}(x,y,z) = x^3\hat{\imath} + y^3\hat{\jmath} + y^3\hat{k}$ outward through unit sphere S is
   (a) 4π (b) $\frac{12\pi}{5}$ (c) $\frac{8\pi}{3}$ (d) None of these.

10. If S_1 and S_2 are smooth oriented surfaces in $\mathbb{R}^3$ having same boundary C and $\bar{F}$ is a vector field on $\mathbb{R}^3$ then $\iint_{S1}(\nabla\times\bar{F}).\hat{n}_1 dS = \iint_{S2}(\nabla\times\bar{F}).\hat{n}_2 dS$ if and only if

(a) $\hat{n}_1 = \hat{n}_2$ (b) $\hat{n}_1 = -\hat{n}_2$ (c) $S_1 \cap S_1 = \emptyset$ (d) None of these.

11. Curl $\left(grad\ (x+z)\right)$ is

a) $x^2\,\hat{\imath} + z^2\hat{\jmath}$. b) 0. c) $\vec{0}$ d) none of these.

12. $div\ \left(curl\ (x^2, yz, \sin z)\right)$ is

a) $2x + z + \cos z$ b) 0. c) $\bar{0}$. d) none of these.

13 . $F(x,y,z) = (3xz,\ -5yz,\ z^2)$ and curl $(pyz^2, 0, qxyz) = F$. Then value of p and q are

a) -1 & 3 b) 1 & -3 c) -1 & -3 d) 1 & 3.

14. Let C be the circle $x^2 + y^2 = 4, z = -3$ oriented counterclockwise. Let $F = (y, xz^3, -zy^3)$ and $I = \oint_C \overline{F}\cdot d\bar{r}$ Then

a) Stoke's theorem is applicable and $I = -112\pi$.
b) Stoke's theorem is applicable to calculate I.
c) Stoke's theorem is not applicable but $I = -112\,\pi$.
d) None of the above.

15. The surface integral $\iint_s \nabla\times F.\ \hat{n}\ dS$ where F is continuously differentiable vector field and S is a closed surface is

a) 0 b) depends on F c) depends on S d) none of these.

16. The line integral $\int_C \bar{r}\cdot d\bar{r}$ where C is a simple closed curve is

a) 0 b) 1 c) depends on C d) none of these.

17. $F(x,y,z) = (y+z, x+z, x+y)$ Then

a) curl $F = 0 = div\ F$ b) $div\ F = 3$ and $curl\ F = 0$.
c) $curl\ F = \bar{0}\ and\ div\ F = 0$. d) none of these.

18. $I = \iiint_V (div\ \hat{n})\ dv$ where V is the volume enclosed by a closed surface S. Then I is

a) surface area of S b) volume V c) 0. d) None of these.

19. The surface integral $\iint_S F\cdot\hat{n}\ dS$ where $\bar{F} = \frac{\bar{r}}{r^3}$; $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ over the surface of the sphere centered at (1, 1, 1) and radius 3 is

a) 1. b) depends on r c) 0. d) None of these.

20. The surface integral $\iint_s (\hat{r}\cdot\hat{n})\ dS$ over a closed surface S with volume V is

a) V b) $3V$ c) 0. d) None of these.

21. The surface integral $\iint_s \; ax\hat{\imath} + by\,\hat{\jmath} + czk.dS$ over the surface of a unit sphere enclosing a volume V is

a) $(a+b+c)\;\; 4\pi$ b) $(a+b+c)V$

c) $(a+b+c)4\pi^2$ d) $\frac{4}{3}\;(a+b+c)$

22. The surface integral $\iint_S \;(x^2+y^2)\,i + (y^2+z^2)j + (x^2+z^2)k$ where S is the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ is

a) –3 b) 3 c) 0 d) None of these.

23. If ϕ is a harmonic function and S is the unit sphere, then the surface integral $\iint_s \;\; grad\,\phi\, dS$

a) does not exist b) is 1

c) is the volume of the unit sphere d) is 0

24. S is vertical cylinder of height 2, with its base a circle of radius 1 on the xy plane, centered at the origin and S includes the disks that close it off top and bottom, then the surface integer $\iint yj$ equals

a) π b) 2π c) $\pi/2$ d) $\pi/4$

25. The surface integral $\iint_s \;\; F.\; dS$ for a constant vector field F and S being a closed surface is

a) a non zero constant b) 0 c) never zero d) None of these

26. A vector field F is tangent to the boundary of a region S in space. Then $\iiint_S \;\; div\;\; FdV$,

a) depends on $\bar{F}$ and S.
b) 0
c) depends only on S
d) Gauss Theorem not applicable.

27. The result $\iint_{S_1} \;(\nabla \times F)\;\cdot ndS = \iint_{S_2} (\nabla \times F)\;\cdot ndS$ where surfaces S_1 and S_2. have common boundary can be prove using

a) Only Gauss theorem and not by Stokes theorem
b) Only Stokes theorem and not by Gauss theorem
c) Neither from Stokes not from Gauss theorem
d) None of the above.

DESCRIPTIVE QUESTIONS

(I) Verify Stoke's Theorem for F defined over S:

1.$F(x,y,z) = 3y\hat{\imath} - xz\hat{\jmath} + yz^2\hat{k}, S$ is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$.

2.$F(x,y,z) = (x^2 - y^2)\,\hat{\imath} + 2xy\hat{\jmath}$ and S is the rectangular lamina in the XY -plane bounded by the lines $x = 0, x = a, y = 0$ and $y = b$.

3.$F(x,y,z) = (2x - y, -yz^2, -y^2z), S$ is the upper half of the sphere; $x^2 + y^2 + z^2 = 1$.

4. $F(x,y,z) = x^4\hat{\imath} - xy\hat{\jmath} + z^2\hat{k}$, where S is the triangle with vertices $(2,0,0)$, $(0,2,0)$ and $(0,0,2)$.

5. $F(x,y,z) = (2xy + z^3)\hat{\imath} - x^2\hat{\jmath} + 3z^2x\hat{k}$, and S is the cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ bounded by the plane $z = 0$ and open at the end $z = h$, $(h \neq 0)$.

(II) Using Stoke's Theorem, evaluate the following surface integrals $\iint_S (curl\ F) \cdot ndS$

1. $F(x,y,z) = (y - z, yz, -xz)$ where S consists of the five faces of the cube $0 \leq x \leq 2, 0 \leq y \leq 2$ and $0 \leq z \leq 2$ not in the XY-plane and n is the outward normal.
2. $F(x,y,z) = x\hat{\imath} + z^2\hat{\jmath} + y^2\hat{k}$ wher S is the plane surface $x + y + z = 1$ lying in the first octant.
3. $F(x,y,z) = (zy, yz, zx)$ where S is the triangular surface with vertices $(0,0,0)$, $(1,0,0)$ and $(0,2,1)$.
4. $F(x,y,z) = (y,z,x)$ and S is the surface of the paraboloid $z = 1 - x^2 - y^2; z \geq 0$.
5. $F(x,y,z) = y^2\hat{\imath} + xy\hat{\jmath} + xz\hat{k}$ where S is the surface of the hemisphere $x^2 + y^2 + z^2 = 1;\ z \geq 0$ and n is the unit normal with a non-negative z compnent.
6. $F(x,y,z) = (x - z, x^3 + yz, -3xy^2)$ where S is the surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ above the plane $z = 0$.
7. $F(x,y,z) = (x^3 - y^3, -xyz, y^3)$ where S is the ellipsoid $x^2 + 4y^2 + z^2 - 2z = 4$ lying above XY-plane.

(III) Using Stoke's Theorem evaluate the line integral $\oint F \cdot dr$:

1. $\oint_C xydx + x^2ydy$ taken around the square C with vertices $(1,0)$, $(-1,0)$, $(0,1)$ and $(0,-1)$.
2. $F(x,y,z) = (x^2y^3, a, z)$ and C is the circle $x^2 + y^2 = r^2$. in the XY-plane .
3. $F(x,y,z) = (yz, zx, xy)$ where C is the curve $x^2 + y^2 = 1, z = y^2$.
4. $F(x,y,z) = (x^2, y^2, z^2)$ and C is the curve of intersection of the cylinder $x^2 + y^2 = 2y$ and the plane $y = z$.
5. $F(x,y,z) = (2z, x, 3y)$ where C is the ellipse, that is the intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 4$.
6. $F(x,y,z) = (4y, 2z, 6y)$ where C is the curve of intersection of $x^2 + y^2 + z^2 = 6z$ and $z = x + 3$.

(IV) Verify Gauss Divergence Theorem

1. $F(x,y,z) = (x^2, y^2, z^2)$ over a unit circle.
2. $F(x,y,z) = (yz, zx, xy)$ over the surface of the sphere $x^2 + y^2 + z^2 = 9$.
3. $F(x,y,z) = (x + y^2)\hat{\imath} - 2x\hat{\jmath} + 2yz\hat{k}$ over the volume of the tetrahedron bounded by the co-ordinate planes and the plane $2x + y + 2z = 6$ in the first octant.
4. $F(x,y,z) = (18z - 12x, 3y)$ over the surface of the cone $z = \sqrt{x^2 + y^2}$ bounded by the plane $z = 4$.
5. $F(x,y,z) = y\hat{\imath} + x\hat{\jmath} + z\hat{k}$ over the cylindrical region $x^2 + y^2 = a^2, z = 0$ and $z = h, (h \neq 0)$.

(V) Using Gauss Divergence Theorem evaluate $\iint_S F \cdot n dS$.

1.$F(x,y,z) = (y - x, z - y, y - x)$ and S is the cube bounded by the planes $x = \pm 1, y = \pm 1, z = \pm 1$

2.$F(x,y,z) = (x^3, y^3, z^3)$ and S is the surface $x^2 + y^2 + z^2 = a^2$

3.$F(x,y,z) = (x^2, y^2, z^2)$ and S is the surface of the sphere $x^2 + y^2 + z^2 = 25$ above the plane $z = 3$.

4.$F(x,y,z) = (6x^2 + 2xy, 2y + x^2 z, 4xy^2)$ and S is the surface of the solid in the first octant bounded by the co-ordinate planes, the cylinder $x^2 + y^2 = 4$ and the plane $z = 4$.

5.$F(x,y,z) = (4x - 2y^2, z^2)$ and S is the region bounded by $y^2 = 4x, x = 1, z = 0, z = 3$.

6.$F(x,y,z) = (x^2, y^2, z^2)$ and S is the surface of the cone $\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 0, (0 \le z \le b)$

7. $F(x,y,z) = (x + y, y + z, z + x)$ and S is the region given by $-4 + x^2 + y^2 \le z \le 4 - x^2 - y^2; 0 \le x^2 + y^2 \le 4$.

Miscellaneous Theory Questions

Practical 7

Unit 1

1. Define the double integral of a bounded function $f : S \to \mathbb{R}$ where $S = [a, b] \times [c, d]$ is a rectangle in $\mathbb{R}^2$. Show with usual notation that $m(b-a)(d-c) \leq \iint_S f \leq M(b-a)(d-c)$

2. Define the triple integral of a bounded function $f : R \to \mathbb{R}$ where $S = [a, b] \times [c, d] \times [e, f]$ with usual notation prove that
$m(b-a)(d-c)(f-e) \leq L(p, f) \leq \iiint_S f\, dV \leq U(p, f) \leq M(b-a)(d-c)(f-e)$.

3. Let $S\{(x, y): a \leq x \leq b, \phi_1 \leq y \leq \phi_2(x)\}$ where $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ are continuous.
   a) Suppose $f: S \to \mathbb{R}$ is such that $f(x, y) \geq 0$ and f is continuous in the interior of S. Prove that $\iint_S f \geq 0$
   b) Prove that $\iint_S f$ gives the volume of the solid with base S, bounded above by the surface $z = f(x, y)$. Show further that $\iint_S 1\ dA$ gives the area of S.
4. Let f be defined and bounded on a rectangle $R = [a, b] \times [c, d]$. Suppose f is integrable over R and for each $y \in [c, d]$,the integral $A(y) = \int_a^b f(x, y) dx$ exists, then show that $\iint_R f$ gives the volume of S where $S = \{(x, y, z) \in \mathbb{R}^3: a \leq x \leq b, c \leq y \leq d, 0 \leq z \leq f(x, y)\}$.

5. Prove that a continuous function is integrable for a rectangular domain in $\mathbb{R}^2$.
(Problems on Integrability of bounded functions having finite number of points of discontinuity)

6. a) State and prove Fubini's Theorem for a rectangular domain in $\mathbb{R}^2$.
   b) Prove Algebra of Integrable functions as a corollary using Fubini's theorem for a Rectangular Domains.

7. a) State the change of variables formula for double integral clearly stating the conditions under which it is valid. Explain further, how will you use it to express the double integral in polar co-ordinates.
   b) State the change of variables formula for triple integral, stating clearly the condition under which it is valid. Express further, how will you use it to express the triple integral in cylindrical co-ordinates (r, θ, z) and spherical co-ordinates $(\rho, \theta, \emptyset)$.
   c) State the change of variable formula for double integral over a rectangular domain and invertible affine transformation.
8. Let U be an open set in $\mathbb{R}^2$ containing the rectangle $[a, b] \times [c, d]$. Suppose $f: U \to \mathbb{R}$ is a continuously differentiable function. Show that $g'(x) = \int_c^d \frac{\partial f}{\partial x}(x, y) dy$ where $g(x) = \int_c^d f(x, y) dy$, $\forall x \in [a, b]$. (This is known as Leibniz Rule for differentiation under integral sign).

. Unit 2

1. Define a parameterized curve in $\mathbb{R}^n$. When do you say that two parameterized curves in $\mathbb{R}^n$ are equivalent? Show that two equivalent parameterized curves have essentially the same image set. Show that the converse of this is not true by considering the curves $\alpha(t) = (\cos t, \sin t)\,; 0 \leq t \leq 2\pi$ and $\beta(t) = (\sin t, \cos t); 0 \leq t \leq 2\pi$.
2. If U is an open set in $\mathbb{R}^2$ /$\mathbb{R}^3$ and Γ is a parameterized curve in U, define the line integral of f along Γ; for a continuous function $f: U \to \mathbb{R}$. Show further, if α, β are two equivalent parameterized curves, then the line integrals of f along them coincide.
3. Let U be an open set in $\mathbb{R}^n$ and $\alpha : [a, b] \to U$ be a parameterization of curve Γ. If $f, g : U \to \mathbb{R}$ are continuous function, then prove that

   $$\int_\Gamma (cf + dg) = c\int_\Gamma f + d\int_\Gamma g$$

   Where c, d are real constants. Further show that $\int_\Gamma f = \int_{\Gamma_1} f + \int_{\Gamma_2} f$, where Γ_1 and Γ_2 are restrictions of α to $[a, c]$ and $[c, d]$ where $a < c < b$.
4. When do you say that two parameterized curves in $\mathbb{R}^n$ are orientedly equivalent? Define the line integral of a vector field F define on an open set U in $\mathbb{R}^n$ along an oriented curve Γ in U. If Γ and Γ' are two orientedly equivalent curves in U, show that $\int_\Gamma F = \int_{\Gamma'} F$.
5. Let f be a continuously differentiable scalar field defined on an open set U in $\mathbb{R}^n$. Suppose P, Q are two points of U that can be connected by piecewise smooth curve C lying in U. Prove that

   $\int_C \nabla f \cdot dr = f(Q) - f(P)$ given that C has parameterization $r(t), t \in [a, b]$ with $r(a) = P$ and $r(b) = Q$.

   OR

   Let f be a continuously differentiable scalar field defined on an open set U in $\mathbb{R}^n$. Suppose C is a closed curve in U, with parameterization $r(t), t \in [a, b]$. Then prove that $\oint_C \nabla F \cdot dr = 0$.
6. Suppose F is a continuous vector field defined on an open connected set U in $\mathbb{R}^n$. Define a function $\phi: U \to \mathbb{R}$ by $\phi(v) = \int_{v_0}^{v} F$ where v_0 is a fixed point in U and F is conservative.

   Show that $\nabla\phi(v) = F(v). \forall v \in U$.
7. Let F be a continuous vector field defined on an open connected set U in $\mathbb{R}^n$. Show that the following conditions are equivalent.
   (i) The line integral of F depends only on the end points of a curve in U and not on the curve.
   (ii) F is the gradient of a C' function (i.e. F has a potential function) on U.
   (iii) For any C' closed curve C in U, $\oint_C F = 0$.
8. State and prove Green's Theorem for a rectangle.
9. State Green's Theorem for a closed region in $\mathbb{R}^2$ whose boundary is a simple closed curve. Show how it can be used to calculate area of the region.
10. $F = (P, Q)$ is a continuously differentiable function defined on a simply connected region D in $\mathbb{R}^2$. Show that $\oint Pdx + Qdy = 0$ around every closed curve C in D if and only if $\frac{\partial p}{\partial y} = \frac{\partial Q}{\partial x}, \forall (x, y) \in D$.

Unit 3

1. Suppose U is an open set in $\mathbb{R}^3$ and $F: U \to \mathbb{R}^3$ be a continuously differentiable vector field.
   Define (i) curl F (ii) $div\ F$.
   Show that necessary and sufficient condition for a C^2 function $F : \mathbb{R}^3 \to \mathbb{R}^3$ to be conservative is that curl $F = 0$.

2. Define a parameterized surface in $\mathbb{R}^3$.When do you say that two parameterized surfaces in $\mathbb{R}^3$ equivalent? If S is a smooth parameterized surface in an open set U in $\mathbb{R}^3$ and $f: U \to \mathbb{R}$ is a continuous scalar field, define the surface integral of $over\ S$. Further if $F : U \to \mathbb{R}^3$ is a continuous vector field, define the surface integral of Fover S.

3. For the surface $\bar{r}(u,v)$ described by the vector equation $\bar{r}(u,v) = X(u,v)i + \quad Y(u,v)j + Z(u,v)\hat{k}, (u,v)u \neq\in T$ where X, Y, Z are differentiable on T, define the fundamental vector product $\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial u}$. If C is a smooth curve lying on the surface, $C = \bar{r}\ (\propto (t)),\ \propto; [a,b] \to T$, then show that $\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial u}$ is normal to C at each point.

4. Let $S = \bar{r}(T)$ be a smooth parametric surface in uv plane. Define area of S. If S is represented by an equation $z = f(x,y)$ then show that area of S is given by

$$\iint_T \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\ dxdy$$

$where\ T$ is projection of S on xy −plane.

5. Let $S = \bar{r}(T)$ be a smooth parametric surface described by a differentiable function $\bar{r}$ defined on region T. Let f be defined and bounded on S. Define surface integral of f over S. If R smoothly equivalent functions, $R(s,t) = \bar{r}\ (G(s,t))$ where $G(s,t) = u(s,t)\hat{\imath} + v(s,t)\hat{\jmath}$ being continuously differentiable. Then show that

$$\iint_{r(A)} f dS = \iint_{R(B)} f d\,S$$

Where $G(B) = A$.

6. State and prove Stoke's Theorem for an oriented smooth, simple parameterized surface in $\mathbb{R}^3$ bounded by a simple, closed curve traversed counter clockwise assuming general form of Green's Theorem.
7. If S and C satisfy hypothesis of Stoke's Theorem and f, g have continuous second order partial derivative. Prove with usual notations.
   a) $\int_C (f\nabla g).d\bar{r} = \iint_S (\nabla f \times \nabla g)\hat{n}dS$
   b) $\int_C (f\ \nabla f).\ d\bar{r} = 0$
   c) $\int_C (f\nabla g) + g\ \nabla f).\ d\bar{r} = 0$
8. State Divergence Theorem for a solid in 3-space (or $\mathbb{R}^3$) bounded by an orientable closed surface with positive orientation and prove the divergence Theorem for cubical region.

9. State and Prove Divergence Theorem for a simple solid region V bounded by an orientable surface S which can be projected on XY, YZ, ZX planes.

10. Prove the following identities, assuming S and V satisfy the conditions of the Divergence Theorem and scalar fields f and g, components of $\bar{F}$ have continuous partial derivatives, $\hat{n}$ is unit outward normal.
   a) $\int_S \bar{A}.\ \hat{n}dS = 0$ Where $\bar{A}$ is a constant vector.
   b) $|V| = \frac{1}{3}\iint_S \bar{r}.\ \hat{n}dS$ where $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $|V|$ = volume of V.
   c) $\iint_S curlF\ .\ \hat{n}dS = 0$.
   d) $\iint_S (D_n f)dS = \iiint_V \nabla^2 f dV$, where $D_n f$ is the direction derivative in the direction $\hat{n}$.
   e) $\iint_S (f\nabla g)\ .\hat{n}dS = \iiint_V (f\nabla^2 g + \nabla f.\ \nabla g)dV$.
   f) $\iint_S (f\nabla g - g\nabla f).\ \hat{n}dS = \iiint_V (f\nabla^2 g - g\nabla^2 f)dV$.

Practical no 2.1. Quotient Space and Orthogonal Transformations,Isometries

Q 1) Let $V=\mathbb{R}^3, W_1=\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1+x_2+x_3=0\}$ and $W_2=\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1-x_2+x_3=0\}$ are subspaces of V. then

(a) $dimV/W_1=dimV/W_2=2, dimW_2/W_1\cap W_2=1$

(b) $dimV/W_1=dimV/W_2=1, dimW_2/W_1\cap W_2=1$

(c) $dimV/W_1=dimV/W_2=1, dimW_2/W_1\cap W_2=2$

(d) None of the above.

Q 2) Let $V=M_2(\mathbb{R})$, $W_1=$ Space of 2×2 real symmetric matrices, $W_2=$ Space of 2×2 real skew symmetric matrices.

(a) $dimV/W_1=1, dimV/W_2=1$ (b) $dimV/W_1=2, dimV/W_2=2$
(c) $dimV/W_1=1, dimV/W_2=3$ (d) None of the above.

Q 3) Let $V=P_2[x]$, the space of polynomial of degree ≤ 2 over $\mathbb{R}$ along with zero polynomial and $W=\{f\in V : f(0)=0\}$. Then

(a) $\{\overline{1},\overline{x+1},\overline{(x+1)^2}\}$ is the basis of the quotient space V/W.

(b) $\{\overline{x+1},\overline{x^2+1}\}$ is the basis of the quotient space V/W

(c) $\{\overline{x+1}\}$ is the basis of the quotient space V/W

(d) None of the above.

Q 4) Let V be a real vector space and $T:\mathbb{R}^6\to V$ be a linear transformation such that $S=\{Te_2,Te_4,Te_6\}$ spans V. Then, which of the following is true ?

(a) S is a basis of V

(b) $\{e_1+KerT, e_3+KerT, e_5+KerT\}$ is a basis of $\mathbb{R}^6/KerT$

(c) $dimV/ImT\geq 3$

(d) $dim\mathbb{R}^6/KerT\leq 3$

Q 5) Consider $W=\{(x,y,z)\in\mathbb{R}^3 : 2x+2y+z=0, 3x+3y-2z=0, x+y-3z=0\}$.Then $dim\mathbb{R}^3/W$ is

(a) 1 (b) 2 (c) 3 (d) 0

Q 6) Consider the linear transformation $T:P_2[\mathbb{R}]\to M_2(\mathbb{R})$ defined by $T(f)=\begin{pmatrix} f(0)-f(2) & 0 \\ 0 & f(1)\end{pmatrix}$ where $P_2[\mathbb{R}]=$ space of polynomials of degree ≤ 2 along with 0 polynomial.Then

(a) $kerT=0$ and $dim(M_2(\mathbb{R})/ImT)=3$

(b) $dim(P_2[\mathbb{R}]/KerT)=1$

(c) T is one-one and onto.

(d) $dim(P_2[\mathbb{R}]/KerT)=2$

Q 7) Let $V = M_2(\mathbb{R})$ and $W = \left\{A \in M_2(\mathbb{R}) : A\begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} A\right\}$.Then

(a) $dim V/W = 0$ (b) $dim V/W = 1$
(c) $dim V/W = 2$ (d) $dim V/W = 3$

Q 8) Let $V = \mathbb{R}^4$ and $W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 \text{ and } x_3 = x_4\}$ a subspace of V. Then

(a) $\{\overline{(1,1,0,0)}, \overline{(0,1,0,1)}\}$ is the basis of V/W.
(b) $\{\overline{(1,0,1,0)}, \overline{(0,-1,0,-1)}\}$ is the basis of V/W
(c) $\{\overline{(1,0,1,0)}, \overline{(0,1,0,1)}\}$ is the basis of V/W
(d) None of the above.

Q 9) Let $V = M_2(\mathbb{R})$.Consider the subspaces $W_1 = \left\{\begin{pmatrix} a & -a \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}\right\}$ and $W_2 = \left\{\begin{pmatrix} a & b \\ -a & d \end{pmatrix} : a, b, d \in \mathbb{R}\right\}$.Then

(a) $dim V/W_1 = dim V/W_2 = 2, dim W_2/W_1 \cap W_2 = 1$
(b) $dim V/W_1 = dim V/W_2 = 1, dim W_2/W_1 \cap W_2 = 1$
(c) $dim V/W_1 = dim V/W_2 = 1, dim W_2/W_1 \cap W_2 = 2$
(d) None of the above.

Q 10) Let $V = M_2(\mathbb{R})$ and $W = \{A \in M_2(\mathbb{R}) : Tr(A) = 0\}$ a subspace of V. Then

(a) $\left\{\overline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \overline{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}\right\}$ is the basis of V/W. (b) $\left\{\overline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}\right\}$ is the basis of V/W
(c) $\left\{\overline{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}\right\}$ is the basis of V/W (d) None of the above.

Q 11) Let $V = P_n[x]$,the space of polynomials of degree$\leq$ n over $\mathbb{R}$ along with zero polynomial and D denote the linear transformation $D : V \to P_{n-1}[x]$ defined by $D(f) = \frac{df}{dx}$. If $W = ker D$, then

(a) $dim V/W = n - 1$. (b) $dim V/W = 1$
(c) $dim V/W = n$ (d) None of these.

Q 12) Let A be a 5×7 matrix over $\mathbb{R}$. Suppose rank $A = 3$.
A linear transformation $T : \mathbb{R}^7 \to \mathbb{R}^5$ is defined by$T(X) = AX$, where X is a column vector in $\mathbb{R}^7$, and $W = ker T$, $U = Img T$, then

(a) $dim \mathbb{R}^7/W = 3, dim \mathbb{R}^5/U = 2$. (b) $dim \mathbb{R}^7/W = 2, dim \mathbb{R}^5/U = 2$.
(c) $dim \mathbb{R}^7/W = 2, dim \mathbb{R}^5/U = 1$. (d) None of the above.

Q 13) Let $V = M_2(\mathbb{R})$ and $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. A linear transformation $T : V \to V$ is defined by $T(B) = AB - B$. Then

(a) T is a linear isomorphism. (b) $dim V/ker T = 1$.
(c) $dim V/ker T = 2$. (d) None of these.

Q 14) Let U, W be vector spaces over $\mathbb{R}$ with bases $\{u_1, u_2, ..., u_m\}$ and $\{w_1, w_2, ..., w_n\}$ respectively. Let $V = U \oplus V$ and linear transformation $P_U : V \to U$ be defined by $P_U(u+v) = u$, where $u \in U$ and $w \in W$. Then

(a) $dim V/ker P_U = n$. (b) $dim V/ker P_U = m$.
(c) $dim V/ker P_U = m - n$. (d) None of these.

Q 15) Let $V = \mathbb{R}^2, W = \{(x, y) \in \mathbb{R}^2 : y = x\}$. Then

(a) $\{\overline{(1,1)}\}$ is a bases of V/W. (b) $\{\overline{(1,0)}\}$ is a bases of V/W.
(c) $\{\overline{(1,1),(1,-1)}\}$ is a bases of V/W. (d) None of the above.

Q 16) If $\alpha : \mathbb{R}^4 \to \mathbb{R}^4$ and $\beta : \mathbb{R}^4 \to \mathbb{R}^4$ are translations such that $\alpha((1,1,1,1)) = (1,0,-1,3)$ and $\beta((2,2,2,2)) = (2,0,3,4)$ then $\alpha\beta(0,0,0,0)$ is
(a) $(0,0,0,0)$. (b) $(0,-3,-1,4)$. (c) $(0,3,1,-4)$. (d) None of these.

Q 17) If $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry defined by $\alpha((x,y)) = (\frac{x}{2} + \frac{\sqrt{3}y}{2} - \frac{1}{2}, \frac{-\sqrt{3}x}{2} + \frac{y}{2} + \frac{\sqrt{3}}{2})$ and $\alpha((x,y)) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$ then
(a) $x = 1, y = -1$. (b) $x = \sqrt{3}, y = 1$. (c) $x = 1, y = 1$. (d) None of these.

Q 18) Let α be an orthogonal transformation of the plane such that the matrix of α w. r. t. the standard basis of $\mathbb{R}^2$ is $\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$, then α represents

(a) a rotation about origin through $\frac{\pi}{4}$. (b) a rotation about origin through $\frac{5\pi}{4}$.
(c) a rotation about the line $y = -x$. (d) None of the above.

Q 19) Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ represents the rotation about origin by angle $\frac{\pi}{4}$ and $\beta : \mathbb{R}^2 \to \mathbb{R}^2$ represents a reflection about y-axis. Then $\beta \circ \alpha$ represents

(a) a rotation about origin through angle $\frac{3\pi}{8}$. (b) a rotation about the line $y = x$.
(c) a rotation about the line $y = -x$. (d) None of the above.

Q 20) Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation and $E = \{v \in \mathbb{R}^3 : \alpha v = v\}$. Then

(a) $dim E = 1$ (b) $dim E \geq 1$
(c) If $dim E = 2$, then α is reflection with respect to the plane.
(d) None of the above.

Q 21) Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ represents reflection in the plane $x + y + z = 0$. The matrix of α with respect to the standard basis of $\mathbb{R}^3$ is
(a) $\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -1 \end{pmatrix}$ (b) $\frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (d) None of these.

Q 22) Let V be an n-dimensional real inner product space. Suppose $B = \{e_i\}_{i=1}^n$ and $B' = \{f_i\}_{i=1}^n$ are orthogonal basis of V. Then

(a) If $T : V \to V$ is a linear transformation such that $T(e_i) = f_i$ for $i = 1$ to n, then T is orthogonal.

(b) If $T : V \to V$ is a linear transformation such that $T(e_i) = f_i$ for $i = 1$ to n, then T need not be orthogonal.

(c) There exist a linear transformation $T : V \to V$ such that $\{T(e_i)\}_{i=1}^n$ is an orthogonal basis of V, but $\{T(f_i)\}_{i=1}^n$ is not an orthonormal basis of V.

(d) None of the above.

Q 23) Let A and B be $n \times n$ real orthogonal matrices. Then

(a) AB and $A + B$ are orthogonal matrices. (b) AB and BA are orthogonal matrices.
(c) $A + B$ is an orthogonal matrix. (d) None of the above.

Q 24) Let A, B be $n \times n$ real matrices. If A and AB are orthogonal matrices,then

(a) B is orthogonal but BA may not be orthogonal (b) B and BA both are orthogonal matrices.
(c) B may not be orthogonal matrix. (d) None of the above.

Q 25) Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry fixing origin and $\alpha \neq$ identity. Then

(a) $\alpha((1, 0))$ is in the first quadrant. (b) $\alpha((1, 0)) \in \{(-1, 0), (0, 1), (0, -1)\}$.
(c) $\alpha((1, 0))$ lies on the unit circle S^1. (d) None of the above.

Q 26) If $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $\langle v, w\rangle = 0 \Rightarrow \langle \alpha(v), \alpha(w)\rangle = 0$ $\forall\ v, w \in \mathbb{R}^2$. Then

(a) α is an isometry of $\mathbb{R}^2$. (b) α is an orthogonal transformation.
(c) $\alpha = aT$ where T is an orthogonal transformation and $a \in \mathbb{R}$. (d) None of the above.

Q 27) Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\alpha((x, y)) = (ax + by + e, cx + dy + f)$ where $a, b, c, d, e, f \in \mathbb{R}$. Then α is an isometry if and only if

(a) $ad - bc \neq 0, e, f > 0$ (b) $ad - bc = \pm 1$.
(c) $a^2 + c^2 = 1$,$b^2 + d^2 = 1, ab + cd = 0$. (d) None of the above.

Q 28) Let V be a finite dimensional inner product space and $\alpha : V \to V$ be an isometry. Then

(a) α is one-one may not be onto. (b) α is one-one only if $\alpha(0) = 0$.
(c) α is bijective. (d) None of the above.

Practical 2.1 Descriptive Questions

Q 1) Let $V = \mathbb{R}^3$ and $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 3x_1\}$ be subspace of V. Find a basis of W and the quotient space V/W.

Q 2) $V = M_2(\mathbb{R})$ and $W = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R} \right\}$. Find the basis of the subspace W and the quotient space V/W

Q 3) Let $V = P_4[x]$,the space of polynomial of degree ≤ 4 along with zero polynomial. The linear transformation $D : P_4[x] \to P_4[x]$ be defined by $D(f) = \frac{df}{dx}$. If $W = kerD$,then find bases of W and V/W.

Q 4) Let $V = M_2(\mathbb{R})$ and N be a 2×2 nilpotent matrix. If $T : V \to V$ is defined by $T(A) = NA - A$, then find $kerT, dimkerT$ and $dimV/kerT$. Is T onto? Justify your answer.

Q 5) Let $P_n[\mathbb{R}]$ denote the space of polynomials with real coefficients of degree $\leq n$ along with zero polynomial.Consider the linear transformation $D : P_n[\mathbb{R}] \to P_{n-1}[\mathbb{R}]$ defined by $D(f) = \frac{df}{dx}$ and $T : P_n[\mathbb{R}] \to P_{n+1}[\mathbb{R}]$ defined by $T(f) = xf$.If $A = DT - TD : P_n[\mathbb{R}] \to P_n[\mathbb{R}]$, find $KerA$, and $dim(A/KerA)$.

Q 6) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T((x_1, x_2, x_3)) = (x_1, x_2)$. Find basis of $kerT$ and $\mathbb{R}^3/kerT$,

Q 7) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}$. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x) = AX(X$ being a column vector in $\mathbb{R}^3)$. Find $kerT$,a basis of $kerT$ and $\mathbb{R}^3/kerT$. Also find ImT.

Q 8) Let $V = M_2(\mathbb{R})$ and $W =$ Space of 2×2 real symmetric matrices. Find a basis of W and the quotient space V/W.

Q 9) Let $V = P_2(\mathbb{R})$, the space of polynomials of degree ≤ 2 over $\mathbb{R}$ along with zero polynomial. A linear transformation $T : P_2(\mathbb{R}) \to \mathbb{R}$ is defined by $T(f) = \int_0^1 f(t)dt$. Find $kerT$ and basis of $kerT, V/kerT$.

Q 10) Show that following maps are isometries.

1. $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\alpha((x, y)) = (x - 2, y + 1)$.
2. $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\alpha((x, y, z)) = (x, -y, z)$.
3. $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\alpha((x, y)) = (\frac{3}{5}x + \frac{4}{5}y - \frac{1}{4}, -\frac{4}{5}x + \frac{3}{5}y + \frac{3}{4})$.
4. $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\alpha((x, y, z)) = (\frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z, x, -\frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z)$.

Q 11) Show that the given maps are orthogonal transformations. Determine whether they are rotations or reflections. In case of rotations,determine the angle of rotation and in case of reflection ,determine the line of reflection.

1. $T(x, y) = (\frac{3}{5}x + \frac{4}{5}y, -\frac{4}{5}x + \frac{3}{5}y)$.
2. $T(x, y) = (\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y)$.
3. $T(x, y) = (-x, -y)$.
4. $(x, y) = (\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y)$

Q 12) Show that the following maps are isometries. Express each of them as a composite of an orthogonal transformation and a translation.

1. $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\alpha(x, y, z) = (\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - 2, \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y + 3, z + 2)$.
2. $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\alpha(x, y) = (\frac{x}{2} - \frac{\sqrt{3}}{2}y + 1, \frac{\sqrt{3}}{2}x + \frac{y}{2} - 5)$.
3. $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\alpha(x, y, z) = (\frac{x}{2} + \frac{\sqrt{3}}{2}z - 1, y, \frac{\sqrt{3}}{2}x - \frac{z}{2} + 5)$.

Q 13) Find the orthogonal transformations in $\mathbb{R}^2$ which represents reflections with respect to the following lines.

1. $x = y$
2. $y = -x$
3. $y = 2x$.

Q 14) Find the orthogonal transformations in $\mathbb{R}^3$ which represent reflections with respect to the following planes.

1. $x - y + z = 0$
2. $2x - y = 0$
3. $y = 0$

Q 15) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $\langle u, v\rangle = 0 \quad \Rightarrow \langle T(u), T(v)\rangle = 0$ for each $u, v \in \mathbb{R}^2$, show that $T = aS$, where $S : \mathbb{R}^2 \to \mathbb{R}^2$ is an orthogonal transformation.

Practical no 2.2. Cayley-Hamilton Theorem and its applications

Q 1) Let $A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$, then

(a) $A^{-1} = \frac{1}{16}[A + 8I]$ (b) $A^{-1} = \frac{1}{16}[A - 8I]$
(c) $A^{-1} = \frac{1}{16}[-A + 8I]$ (d) $A^{-1} = \frac{1}{16}[-A - 8I]$

Q 2) The following pairs of n x n matrices do not have same characteristic polynomial.

(a) A and A^t. (b) A and PAP^{-1} where P is non singular $n \times n$ matrix.
(c) A and A^2. (d) AB and BA.

Q 3) Let $p(t) = t^2 + bt + c$ where $b, c \in \mathbb{R}$.Then the number of real matrices having $p(t)$ as characteristic polynomial is

(a) One (b) Two
(c) Infinity (d) None of the above

Q 4) Let $p(t) = t^3 - 2t^2 + 5$ be the characteristic polynomial of A then $\det A$ and $\operatorname{tr} A$ are

(a) $5, -2$ (b) $2, 5$
(c) $-5, 2$ (d) $-2, 5$

Q 5) If A is a 3×2 matrix over $\mathbb{R}$ and B is a 2×3 matrix over $\mathbb{R}$ and $p(t)$ is the characteristic polynomial of AB, then

(a) t^3 divides $p(t)$ (b) t^2 divides $p(t)$
(c) t divides $p(t)$ (d) None of the above

Q 6) Let A and B be $n \times n$ matrix over $\mathbb{R}$ such that $\operatorname{tr} A = \operatorname{tr} B$ and $\det A = \det B$.Then

(a) Characteristic polynomial of $A =$ Characteristic polynomial of B.

(b) Characteristic polynomial of $A \neq$ Characteristic polynomial of B.

(c) Characteristic polynomial of $A =$Characteristic polynomial of B if $n = 3$.

(d) Characteristic polynomial of $A =$Characteristic polynomial of B if $n = 2$.

Q 7) Let A and B be $n \times n$ matrix over $\mathbb{R}$ such that characteristic polynomial of $A =$ characteristic polynomial of B.Then

(a) A and B are similar matrices (b) $\det A = \det B$
(c) $AB = BA$ (d) None of the above.

Q 8) Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Q 9) Let $p(t) = t^3 - 2t^2 + 15$ be the characteristic polynomial of A .Then det A

(a) 15 (b) -15 (c) 0 (d) None of these (a) $A^{10} = \begin{pmatrix} 2^{10} & -2^{10} \\ -2^{10} & 2^{10} \end{pmatrix}$

(b) $A^{10} = \begin{pmatrix} 2^{11} & -2^{11} \\ -2^{11} & 2^{11} \end{pmatrix}$

(c) $A^{10} = \begin{pmatrix} 2^{9} & -2^{9} \\ -2^{9} & 2^{9} \end{pmatrix}$ (d) $A^{10} = \begin{pmatrix} -2^{9} & 2^{9} \\ 2^{9} & -2^{9} \end{pmatrix}$

Q 10) Let A be a 3×3 matrix and λ_1, λ_2 be only two distinct eigen values of A.Then its characteristics polynomial $k_A(x)$ is

(a) $(x-\lambda_1)(x-\lambda_2)$

(b) $(x-\lambda_1)(x-\lambda_2)^2$

(c) $(x-\lambda_1)^2(x-\lambda_2)$

(d) $(x-\lambda_1)^2(x-\lambda_2)$ or $(x-\lambda_1)(x-\lambda_2)^2$

Q 11) Let characteristic polynomial of A is $t^2+a_1t+a_0$ and and characteristic polynomial of A^{-1} is $t^2+a_1^{'}t+a'_0$.Then

(a) $a_0a_0^{'}=1$ and $a_1+a_1^{'}=1$ (b) $a_1a_1^{'}=1$ and $a_0a_0^{'}=1$
(c) $a_0a_0^{'}=1$ (d) $a_0a_0^{'}=1$ and $a_1^{'}=a_1a_0^{'}$

Q 12) If $p_1(t)=t^2+a_1t+a_0$ is characteristic polynomial of A and $p_2(t)=t^2+a_1^{'}t+a_0^{'}$ is characteristic polynomial of A^2 then

(a) $a_1^{'}=a_1^2$ and $a_0^{'}=a_0^2$ (b) $a_1^{'}=2a_1$ and $a_0^{'}=a_0^2$
(c) $a_0^{\prime}=a_0^2,\ a_1^{\prime}=a_1^2-2a_0$ (d) None of the above

Q 13) Let $A_{6\times 6}$ be a matrix with characteristic polynomial $x^2(x-1)(x+1)^3$, then trace A and determinant of A are

(a) -2, 0 (b) 2, 0 (c) 3, 1 (d) 3, 0

Q 14) $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ are similar (non- zero a, b, d)

(a) for any reals a, b, d. (b) if $a = d$.
(c) if $a \neq d$. (d) never similar.

Q 15) Let $A_{6\times 6}$ be a diagonal matrix over $\mathbb{R}$ with characteristic polynomial $(x-2)^4(x+3)^2$. Let $V = \{B \in M_6(\mathbb{R}) : AB = BA\}$. Then dim $V =$

(a) 8 (b) 12 (c) 6 (d) 20.

Q 16) If $A - I_n$ is a $n \times n$ nilpotent matrix over $\mathbb{R}$, then characteristic polynomial of A is

(a) $(t-1)^n$ (b) t^n
(c) $t^n - 1$ (d) $(t^{n-1}-1)t$

Q 17) If $A \in M_2(\mathbb{R})$, $tr\, A = -1$, $det\, A = -6$ then $det\,(I_2 + A)$ is

(a) -6 (b) -5 (c) -1 (d) None of the above.

Q 18) Let $A = [a_{ij}]_{10\times 10}$ be a real matrix such that $a_{i,i+1} = 1$ for $1 \leq i \leq 9$ and $a_{ij} = 0$ otherwise, then

(a) $A^9(A-I)$ (b) $(A-I)^{10}$ $A^{10}=0$ $A(A-I)^9=0$

Q 19) $T : \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation such that $T^3 + 3T^2 = 4I$. If $S = T^4 + 3T^3 - 4I$, then

(a) S is not one-one. (b) S is one-one.

(c) if 1 is not an eigen value of T then S is invertible.

(d) None of these.

Q 20) Which of the following statements are true

1. If the characteristic roots of two $n \times n$ matrices are same then their characteristic polynomials are same.
2. If the characteristic polynomials of two $n \times n$ matrices are same then their characteristic roots are same.
3. If eigen values of two $n \times n$ matrices are same then their eigen vectors are same.
4. The characteristic roots of two $n \times n$ matrices are same but their characteristic polynomials may not be same.

(a) ii and iv are true. (b) i, iii are true.
(c) i, ii and iii are true. (d) only ii is true.

Q 21) A 2×2 matrix A has the characteristic polynomial $x^2 + 2x - 1$, then the value of det $(2I_2 + A)$ is
(a) $\frac{1}{det\, A}$ (b) 0
(c) $2 + det\, A$ (d) $2\, det\, A$

Practical 2.2 Descriptive Questions

Q 1) Let $A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 0 & 2 \\ -1 & 2 & 4 \end{pmatrix}$. Then show that 5 is a characteristic root of A.

Q 2) Find characteristic polynomial of the following matrix and verify Cayley-Hamilton theorem for $A = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{pmatrix}$.

Q 3) Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

(a) Find characteristic polynomial of A.

(b) Using Cayley-Hamilton theorem find A^{10}.

Q 4) Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{pmatrix}$

(a) Find characteristic polynomial of A.

(b) Using Cayley-Hamilton theorem,find A^{-1} and $A^4 - 3A^3 - 3A^2 + 7A + 6I$.

Q 5) Let $A = \begin{pmatrix} 6 & -5 \\ 2 & -1 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 5 \\ 1 & 2 \end{pmatrix}$

(a) Find $P^{-1}AP$ and compute A^6

(b) Verify that A and $P^{-1}AP$ have same characteristic polynomial.

Q 6) Let $A = \begin{pmatrix} 10 & -9 & 0 \\ 4 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

(a) Find characteristic polynomial of A and express it as a product of linear factors.

(b) Compute A^{-1} using Cayley-Hamilton theorem and find characteristic polynomial of A^{-1} and factorise it.

(c) Verify that product of constant terms of both the polynomials is 1.

Q 7) (a) If $A = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ Find A^{-1} and characteristic polynomial of A^{-1}.

(b) If $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1. \end{pmatrix}$. Find A^{-1}.

Q 8) (a) Verify Cayley-Hamilton theorem for $\begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$. Find A^{-1} if it exist.

(b) Verify Cayley-Hamilton theorem for $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Find A^{-1} and $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

Q 9) (a) $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$. Find $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

(b) $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$. Find $A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I$.

Q 10) (a) Find A^n for $A = \begin{pmatrix} 7 & 3 \\ 2 & 6 \end{pmatrix}$.

(b) Show that$A^n = A^{n-2} + A^2 - I$ for $n > 3$ for $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Hence write A^{50}.

Q 11) Let $P(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be a monic polynomial of degree 3 where $a_0, a_1, a_2, a_3\mathbb{R}$ and $A = \begin{pmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & \cdots & 1 & -a_3 \end{pmatrix}$, then show that characteristic polynomial of A is $P(x)$.

Q 12) Let $A_{7\times 7}$ be a diagonal matrix over $\mathbb{R}$ with characteristic polynomial $(t+4)^3(t-3)^4$. Let $V = \{B \in M_7(\mathbb{R}) : AB = BA\}$. Find dim V.

Q 13) Express the characteristic polynomial of $aI + bA$ interms of the characteristic polynomial of A.

Q 14) Let A and C be matrices such that $ACA = 0$, show that for every matrix B the characteristic polynomial of AB and $A(B + C)$ are equal.

Q 15) If $A = uv^t$, then the characteristic polynomial of A is $x^{n-1}(x - u^t v)$ where u, v are column vectors in $\mathbb{R}^n$.

Practical no 2.3. Eigen values and Eigen vectors

Q 1) The product of all characteristic roots of a square matrix A is equal to
(a) 0 (b) 1 (c) $|A|$ (d) None of these.

Q 2) If eigen value of A is λ,then eigen value of A^2 is
(a) 1 (b) $\frac{1}{\lambda}$ (c) λ^2 (d) None of these.

Q 3) If A is invertible matrix and eigen value of A is λ, then eigen value of A^{-1} is
(a) 1 (b) $\frac{1}{\lambda}$ (c) λ (d) None of these.

Q 4) If the determinant of a matrix A is non-zero, then its eigen values of A are
(a) 1 (b) 0 (c) Non-zero (d) None of these.

Q 5) If the determinant of a matrix A is zero, then one of its eigen values of A is
(a) 1 (b) 0 (c) -1 (d) None of these.

Q 6) The eigen space corresponding to eigen value 1 of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has basis
(a) $\{(1,0)\}$ (b) $\{(1,0),(0,1)\}$
(c) $\{(0,1)\}$ (d) $\{(1,1)\}$

Q 7) Let $A = \begin{bmatrix} a & b & 1 \\ c & d & 1 \\ 1 & -1 & 0 \end{bmatrix}$ where $a, b, c, d \in \mathbb{R}$ such that $a + b = c + d$, then A has eigen value
(a) $a + c$ (b) $a + b$ (c) $a - d$ (d) $b - d$

Q 8) Zero is a eigen value of a linear map T from V to V if and only if
(a) Ker$T = \{0\}$ (b) T is bijective
(c) T is singular (d) T is non singular

Q 9) The eigen values of a 3×3 real matrix A are 1,2,3.Then

(a) Inverse of A exists and it is $\frac{1}{6}(5I + 2A - A^2)$
(b) Inverse of A exists and it is $\frac{1}{6}(5I + 2A + A^2)$
(c) Inverse of A does not exist
(d) None of the above

Q 10) The matrix $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}$ has

(a) Only one distinct eigen value
(b) Only two distinct eigen values
(c) Three distinct eigen values
(d) None of the above

Q 11) The eigen vectors of the matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ generate

(a) a vector space with basis $\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$

(b) a vector space with basis $\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$

(c) a vector space with basis $\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$

(d) a vector space with basis $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$

Q 12) The eigen vectors of the matrix $A=\begin{pmatrix}2&0\\0&2\end{pmatrix}$ generate a vector space of dimension

(a) 1 (b) 2 (c) 3 (d) 4

Q 13) The eigen space $E(5)$ of the matrix $A=\begin{pmatrix}1&4\\2&3\end{pmatrix}$ corresponding to the eigen value $\lambda=5$

(a) is $\begin{pmatrix}1\\1\end{pmatrix}$ (b) is $\begin{pmatrix}2\\-1\end{pmatrix}$

(c) has a basis $\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$ (d) has a basis $\left\{\begin{pmatrix}2\\-1\end{pmatrix}\right\}$

Q 14) Let V a vector space over R and $I\colon V\to V$ be the identity map.Then

(a) v is the only eigen vector of I for some $v\in V$

(b) $2v$ is the only eigen vector of I for some $v\in V$

(c) $3v$ is the only eigen vector of I for some $v\in V$

(d) every vector in V is an eigen vector of I

Q 15) Let $T\colon R^2\to R^2$ be the linear map which rotates every vector $v\in R^2$ through an angle $\frac{\pi}{4}$.Then T has

(a) no eigen vectors

(b) only two eigen vectors

(c) only three eigen vectors

(d) infinitely many eigen vectors

Q 16) Let $A_{3\times3}$ be a real matrix of rank 1, then the eigen values of A are

(a) $0,0,1$ (b) $0,0,tr\,A$ (c) $0,0,det\,A$ (d) $0,0,-det\,A$

Q 17) Let $A=[a_{ij}]$ be a 10×10 matrix with $aij=\begin{cases}1 & \text{if } i+j=11\\ 0 & \text{otherwise}\end{cases}$. Then the set of eigen values of A is

(a) $\{0,1\}$ (b) $\{1,-1\}$ (c) $\{0,1,10\}$ (d) $\{0,11\}$

Q 18) Let $A_{n\times n}$ be a real matrix, then

(a) A,A^t have same determinant, same eigen values and same eigen vectors.

(b) A, A^t have same determinant, same eigen values but eigen vectors may be different.

(c) A, A^t have same eigen values but different determinants.

(d) A, A^t have different eigen values.

Q 19) Let $\sum_{j=1}^{n} a_{ij} = 1$ for a real matrix $A = [a_{ij}]$ then

(a) $(1, 1, \cdots, 1)$ is an eigen vector of A corresponding to the eigen value 1.

(b) $(1, 0, \cdots, 0)$ is an eigen vector of A corresponding to the eigen value 1.

(c) $(1, 1, \cdots, 1)$ is an eigen vector of A corresponding to the eigen value n.

(d) 1 is not an eigen value of A.

Q 20) Let the characteristic polynomial of $A_{3\times3}$ be $x(x-1)(x+2)$, then the characteristic polynomial of A^2 is
(a) $x(x+1)(x-2)$ (b) $x(x-1)(x-4)$
(c) $x(x+1)(x+4)$ (d) None of these.

Q 21) If matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ a & 1 & b \\ 1 & 0 & 0 \end{bmatrix}$ has linearly independent eigen vectors corresponding to eigen value 1, then
(a) $a = 0, b = 0$. (b) $a = 1, b = 1$
(c) for any a, b. (d) $a + b = 0$.

Q 22) Let characteristic polynomial of $A_{2\times2}$ be a real matrix and its characteristics polynomial is $x^2 - 3x + 2$. Then the characteristic polynomial of A^{-1} is
(a) $x^2 - \frac{3}{2}x + \frac{1}{2}$ (b) $x^2 - 3x + 2$

(c) $x^2 - 2x + 3 = 0$ (d) $x^2 - \frac{1}{2}x + \frac{3}{2}$

Q 23) One of the eigen vectors of the matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ over R is

(a) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (d) None of these.

Q 24) If A is a square matrix of order n and λ is a scalar,then the characteristic polynomial of A is obtained by expanding the determinant:
(a) $|\lambda A|$ (b) $|\lambda A - I_n|$ (c) $|A - \lambda I_n|$ (d) None of these

Q 25) At least one characteristic roots of every singular matrix is equal to
(a) 1 (b) -1 (c) 0 (d) None of these.

Q 26) The characteristic roots of two matrices A and BAB^{-1} are
(a) The same (b) Different (c) Always zero (d) None of these.

Q 27) The scalar λ is a characteristic root of the matrix A if:
(a) $A - \lambda I$ is non-singular (b) $A - \lambda I$ is singular (c) A is singular (d) None of these.

Q 28) If eigen value of A is λ,then eigen value of $P^{-1}AP$ is

(a) 1 (b) $\frac{1}{\lambda}$ (c) λ (d) None of these.

Q 29) If λ is a characteristic root of a matrix A then characteristic roots of $-A$ and $\alpha I - A$ respectively are

(a) $-\lambda$ and $\alpha - \lambda$ (b) $-\lambda$ and α (c) $-\lambda$ and λ (d) None of these.

Q 30) Which of the following statements are true

1. If the characteristic roots of two $n \times n$ matrices are same then their characteristic polynomials are same.
2. If the characteristic polynomials of two $n \times n$ matrices are same then their characteristic roots are same.
3. If eigen values of two $n \times n$ matrices are same then their eigen vectors are same.
4. The characteristic roots of two $n \times n$ matrices are same but their characteristic polynomials may not be same.

(a) ii and iv are true. (b) i, iii are true.
(c) i, ii and iii are true. (d) only ii is true.

Q 31) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal transformation of rotation through angle θ, then

(a) T has no eigen values for any $\theta \in (0, 2\pi)$.

(b) T has only one eigen value -1 for $\theta = \pi$ and no eigen values if $\theta \in (0, 2\pi) - \{\pi\}$.

(c) T has eigen value 1 for $\theta = \pi/4$.

(d) T has only one eigen value for all $\theta \in (0, 2\pi)$.

Q 32) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal transformation of reflection in the line $y = \tan\frac{\theta}{2}x$, then

(a) T has no eigen value for any $\theta \in (0, 2\pi)$.

(b) T has only one eigen value 1 for every $\theta \in (0, 2\pi)$.

(c) T has two eigen values $1, -1$ for every $\theta \in (0, 2\pi)$.

(d) T has an eigen value -1 for $\theta = \pi$.

Q 33) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in \mathbb{Z}$ such that $a + b = c + d$, then

(a) A has two integer eigen values.

(b) A may not have any eigen value.

(c) A has two eigen values which may not be integers.

(d) A has two eigen values only if $b, c = 0$.

Q 34) Let A be an $n \times n$ orthogonal matrix with $detA = -1$. Then

(a) -1 is the only eigenvalue of A. (b) -1 is an eigenvalue of A.
(c) A has at least one real eigenvalue only if n is odd. (d) None of the above.

Q 35) Let A be an 2×2 orthogonal matrix with $det A = 1$. Then

(a) 1 is the eigenvalue of A. (b) -1 cannot be an eigenvalue of A.
(c) A may not have real eigenvalue. (d) None of the above.

Q 36) Let $x(x-1)(x+2)$ be the characteristic polynomial of a 3×3 matrix A, then the characteristic polynomial of A^2 is

(a) $x(x-1)(x-4)$ (b) $x(x+1)(x-2)$
(c) $x(x+1)(x+4)$ (d) None of these.

Q 37) Which of the following statements are true-

(i) 0 is an eigen value of a matrix if and only if the matrix is singular.

(ii) $A_{n \times n}$ has atleast one (real) eigen value if n is odd.

(iii) A matrix with all the diagonal entries equal to zero has zero eigen value.

(iv) $det\, A =$ product of characteristic roots of A.

(a) all the statements are true. (b) (i), (ii), (iv) are true.

(c) (i), (iii) are true. (d) (i), (ii), (iii) are true.

Practical 2.3 Descriptive Question

Q 1) Find eigen values and bases of the corresponding eigen spaces for following matrices

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \quad \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix}$$

Q 2) Find eigen values and bases of the corresponding eigen values of $A = \begin{pmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{pmatrix}$.

Hence find eigen values and the bases of corresponding eigen spaces of
(i) A^{-1} (ii) $A^3 - 2A^2 + I$ (iii) $A + 2I$

Q 3) Let A, B be $n \times n$ matrices and x be an eigen vector corresponding to a non-null eigen value λ, of AB. Show that Bx is an eigen vector of BA corresponding to λ.

Q 4) If λ is an eigen value of $A_{n \times n}$, show that the eigen subspace of A^k corresponding to λ^k contains the eigen subspace of A corresponding to λ.

Q 5) Let v be a non-zero vector in $\mathbb{R}^n$ and $A = v^t v$ where v is treated as a $1 \times n$ row vector. Then

a. Show that the eigen values of A are 0 and vv^t.

b. Show that the eigen space corresponding to eigen value vv^t is of dimension 1.

c. Identify the eigen subspace corresponding to 0.

Q 6) Find all the eigen values and corresponding eigen vectors of the matrix

$$(1)\ \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} \qquad (2)\ \begin{bmatrix} 5 & 4 \\ 5 & 6 \end{bmatrix} \qquad (3)\ \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$(4)\ \begin{bmatrix} 3 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad (5)\ \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \quad (6)\ \begin{bmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 6 \end{bmatrix}$$

Q 7) Let matrix $A = \begin{bmatrix} -1 & 2 & 1 \\ -4 & 5 & 1 \\ -1 & -2 & -3 \end{bmatrix}$ and matrix $B = 3A^2 - 4A + I$, where I is 3×3 identity matrix. Find the eigen values and corresponding eigen vectors of the matrix B.

Q 8) Compute the eigen values and corresponding eigen vectors of A^{-1} ,for $A = \begin{bmatrix} 4 & 2 & 5 \\ 3 & 3 & 5 \\ 3 & 2 & 6 \end{bmatrix}$.Hence compute the eigen values and eigen vectors of $4A^{-2} - 3A^{-1} + 2I$.

Q 9) Let $A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & -1 & 3 \end{bmatrix}$.Find the eigen values and corresponding eigen vectors of the matrix B ,where $B = 2A^2 + 3A - 5I$ where I is 3×3 identity matrix.

Q 10) Find eigen values and bases of the corresponding eigen vectors for a 3×3 matrix having all its entries 1.

Q 11) Let V be a vector space of dimension 3 and $\{v_1, v_2, v_3\}$ be a basis of V. Find eigen values and corresponding eigen spaces of $T : V \to V$ be defined by $T(v_1) = v_1, T(v_2) = v_1 + v_2, T(v_3) = v_1 + v_2 + v_3$.

Q 12) If A is a nilpotent matrix ($A^k = 0$ for some $k \in \mathbb{N}$) then show that 0 is the only eigen value of A, hence show that x^n is the characteristic polynomial of A.

Q 13) Find a 3×3 real matrix A s.t. $Au_1 = u_1, Au_2 = 2u_2, Au_3 = 3u_3$ where $u_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, u_2 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$.

Q 14) Let A, B be $n \times n$ real matrices and $P_B(x)$ be the characteristic polynomial of B. Show that the matrix $P_B(A)$ is invertible if and only if A and B have no common eigen values.

Q 15) Let λ_1 and λ_2 be two distinct eigen values of a matrix A and let u_1 and u_2 be eigen vectors of A corresponding to λ_1 and λ_2 respectively, then show that $u_1 + u_2$ is not an eigen vector of A.

Q 16) Prove that if every non-zero vector of $\mathbb{R}^n$ is an eigen vector of $A_{n\times n}$ then A is a $n \times n$ scalar matrix.

Q 17) If $\mathbb{P}_3(\mathbb{R})$ is a vector space containing polynomials of degree ≤ 3 along with the zero polynomial over $\mathbb{R}$ and $D : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R})$ is defined as $D(f(x)) = f'(x)$ then find the characteristic polynomial, eigen values and corresponding eigen space for D.

Q 18) Let $A_{13\times 13}$ be a real matrix of rank 1 and $P(t)$ be the characteristic polynomial of A. Prove or disprove: (i) $t^{12}|P(t)$ (ii) tr A is an eigen value of A.

Q 19) Consider the linear transformation $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $T(A) = A^t$. Find the eigen values and corresponding eigen vectors of T.

Q 20) Prove that $\begin{pmatrix} 1 & 1.00001 & 1 \\ 1.00001 & 1 & 1.00001 \\ 1 & 1.00001 & 1 \end{pmatrix}$ has one positive and one negative eigen value.

Q 21) Let $A = [a_{ij}]$ be a 3×3 real matrix where each $a_{ij} \geq 0$ and $\sum_{j=1}^{3} a_{ij} = 1$ then prove that any eigen value of A has absolute value ≤ 1.

Q 22) Find a 3×3 matrix A which has eigen values 0, 1, -1 with corresponding eigen vectors $(0, 1, -1)^t, (1, -1, 1)^t$, and $(0, 0, 1)^t$ respectively.

Practical no 2.4. Similar matrices and Minimal polynomial

Q 1) If A and B are 3×3 matrices over R having $(1,-1,0)^t, (1,1,0)^t$, and $(0,0,1)^t$ as eigenvectors. Then

(a) A and B are similar matrices. (b) $AB = BA$.
(c) A and B have same eigenvalues. (d) None of the above.

Q 2) If $n \times n$ real matrices A, B are similar and $f(x)$ is a polynomial in real coefficients then $f(A), f(B)$ have

(a) same characteristic polynomials but different minimal polynomials.

(b) same minimal polynomial but different characteristic polynomials.

(c) same characteristic polynomial and same minimal polynomial.

(d) characteristic polynomials are different as well as the minimal polynomials are different.

Q 3) For square matrices A, B of same size, which of the following statements are true?

i. If A, B are similar then they have same characteristic polynomial.

ii. If A, B are similar then they have same eigen vectors.

iii. If A, B have same characteristic polynomial then A, B are similar.

iv If A, B have same characteristic roots then A, B are similar.

(a) i and iv (b) only i

(c) i, ii and iv (d) None.

Q 4) The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is

(a) similar to $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ (b) similar to $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

(c) similar to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) not similar to any diagonal matrix

Q 5) The matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is similar to the matrix

(a) $\begin{pmatrix} 10 & -12 \\ 4 & -5 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 2 \\ 5 & -4 \end{pmatrix}$

(c) $\begin{pmatrix} 6 & 4 \\ 2 & 1 \end{pmatrix}$

(d) None of the above

Q 6) Degree of the minimal polynomial of $n \times n$ real matrix is

(a) equal to n. (b) less than or equal to n.

(c) greater than n. (d) less than n.

Q 7) Minimal polynomial of $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where A, B are square matrices, is

(a) L.C.M. of the minimal polynomials of A and B.

(b) G.C.D. of the minimal polynomials of A and B.

(c) product of the minimal polynomials of A and B.

(d) minimal polynomial of $A-$ minimal polynomial of B.

Q 8) Let $A =$ diag $\{1, 2, -1\}$, $B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, then

(a) B, C, D are similar to A. (b) Only D are similar to A.

(c) None of B, C, D are similar to A. (d) A is similar to D.

Q 9) If A is a square matrix with all its eigen values equal to 1, then

(a) A^k is similar to A for every positive integer k.

(b) A^k is not similar to A for any positive integer $k \neq 1$.

(c) A^k is similar to A for only $k = 2$.

(d) $A^k = I$ for some positive integer k.

Q 10) The minimal polynomial of the diagonal matrix $A =$ diag $\{1, -1, 1, -1\}$ is

(a) $x^2 + 1$ (b) $x^2 - 1$

(c) $(x^2 - 1)^2$ (d) None of these.

Q 11) Let $A_{n \times n}$ be a real matrix, then the characteristic polynomial of $A =$ the minimal polynomial of A if

(a) and only if A has n distinct characteristic roots.

(b) A has n distinct characteristic roots.

(c) only if A is a diagonal matrix. (d) A is nilpotent matrix.

Q 12) The minimal polynomial of $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ is

(a) $x - 1$ for any $\alpha \in \mathbb{R}$. (b) $(x - 1)^2$ for any $\alpha \in \mathbb{R}$.

(c) $x - 1$ if $\alpha = 0$ and $(x - 1)^2$ otherwise. (d) $x - 1$ if $\alpha \neq 0$ and $(x - 1)^2$ otherwise.

Q 13) The minimal polynomial of $\begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 2 \end{bmatrix}$ is

(a) $(x-1)(x-2)$ for any $\alpha, \beta, \gamma \in \mathbb{R}$.

(b) $(x-1)^2(x-2)$ for any $\alpha \in \mathbb{R}$.

(c) $(x-1)^2(x-2)$ if $\alpha = 0$ and $(x-1)(x-2)$ otherwise.

(d) $(x-1)^2(x-2)$ if $\alpha \neq 0$ and $(x-1)(x-2)$ otherwise.

Q 14) If $a = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix}$ then which of the following statements is true

(i) $x - 1$ is the minimal polynomial of A if and only if $\alpha = \beta = \gamma = 0$.

(ii) $(x-1)^2$ is the minimal polynomial of A if and only if $\alpha = \gamma = 0$ and $\beta \neq 0$.

(iii) $(x-1)^3$ is the minimal polynomial of A if and only if β and exactly one of the α, γ are 0.

(iv) $(x-1)^3$ is the minimal polynomial of A if and only if exactly two of the α, β, γ are 0.

(a) i, ii, iii are true. (b) only i is true.

(c) i and iii are true. (d) i, ii, iv are true.

Q 15) Let $A = \begin{bmatrix} 2 & 0 & 0 \\ a & 2 & 0 \\ b & c & -1 \end{bmatrix}$. Then $(t+1)(t-2)$ is the minimal polynomial of A if and only if

(a) $b = c = 0$ (b) $a = 0$

(c) $b \neq 0$ (d) $a = b = c$.

Q 16) If N_1, N_2 are real nilpotent matrices , then N_1, N_2 are similar if and only if

(a) they have same characteristic polynomials. (b) They have same minimal polynomials.

(c) Either N_1 or N_2 is zero. (d) $N_1 = \pm N_2$

Practical 2.4 Descriptive Question

Q 1) Determine the minimal polynomials of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Show that their minimal polynomials are different though their characteristic polynomials are same.

Q 2) Show that a $n \times n$ matrix A such that $A^2 = 0$ is either a zero matrix or is similar to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Q 3) Find the minimal polynomial of

(1) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ (2) $\begin{pmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ 2 & 1 & 0 \end{pmatrix}$

(3) $\begin{pmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ 2 & 1 & 0 \end{pmatrix}$ (4) $\begin{pmatrix} 3 & 1 & 6 \\ 2 & 1 & 0 \\ -1 & 0 & -3 \end{pmatrix}$

Q 4) Find the values of 'a' and 'b' such that the following matrices are similar.

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 2 & a & 2 \\ 3 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & b \end{bmatrix}$$

Q 5) Prove or disprove: $A_{n\times n}$ is idempotent ($A^2 = A$) if and only if the minimal polynomial of A is $x^2 - x$.

Q 6) For every $n \times n$real matrix A, show that there exists an unique monic polynomial of least degree that annihilates A.

Q 7) If $d_1, d_2, \cdots, d_k$ are the distinct diagonal entries of a $n \times n$ diagonal matrix A, then show that the minimal polynomial of A is $\prod_{i=1}^{k}(x - d_i)$.

Q 8) Show that the minimal polynomial of $A_{n\times n}$ is product of k distinct linear factors if and only if A is similar to a diagonal matrix with k distinct diagonal entries.

Q 9) Show that the minimal polynomial of the companion matrix corresponding to $f(x) = x^3 + a_2x^2 + a_1x + a_0$ is $f(x)$.

Q 10) If $A_{n\times n}$ is a real matrix, $A = [a_{ij}]$ where $a_{ij} = \begin{cases} 1 & i+j=n \\ 0 & \text{otherwise.} \end{cases}$. Find the minimal polynomial of A.

Q 11) Let A be a 3×3 matrix with all its entries $= 1$. Find the minimal polynomial of A.

Q 12) find the minimal polynomial of $A = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{bmatrix}$, where $a, b, c \in \mathbb{R}$.

Q 13) Show that characteristic polynomial of $A_{n\times n}$ where $A^2 = A$ and rank $A = k$ is $x^{n-k}(x-1)^k$.

Practical no 2.5. Diagonalization of a matrix

1. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$. Then,
   (a) A and A^{100} are both diagonalizable. (b) A is diagonalizable but A^{100} is not.
   (c) Neither A nor A^{100} is diagonalizable. (d) None of the above.

2. Let $A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix}$ and $B = A^{100} + A^{20} + I$. Then,
   (a) A, B are not diagonalizable. (b)A is diagonalizable, but B is not diagonalizable.
   (c) AB is diagonalizable (d) None of the above.

3. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(61, 23) = (189, 93)$ and $T(67, 47) = (195, 117)$. Then
   (a) T is diagonalizable with distinct eigenvalues. (b) T is not diagonalizable.
   (c) T does not have distinct eigenvalues, but is diagonalizable. (d) None of the above.

4. Which of the following matrices is not diagonalizable?
   (a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

5. Let A be a $n \times n$ real orthogonal matrix. Then
   (a) A has n real eigen values and each eigen value is ± 1. (b) A is diagonalizable
   (c) A may not have any real eigen value. (d) (b) $A^2 = I$

6. Let $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$, then A is diagonalizable if
   (a) $a = b, c = 1$ (b) $a = 1 = b = c$ (c) $a = b = c = 0$ (d) $a, b, c > 0$

7. Let $A = \begin{bmatrix} 0 & a \\ 0 & -a \end{bmatrix}$
   (a) A is diagonalizable but not orthogonally diagonalizable.
   (b) A is not diagonalizable for any $a \in \mathbb{R}$.
   (c) A is orthogonally diagonalizable if and only if $a = 1$ (d) None of these.

8. If A is a 4×4 matrix having all diagonal entries 0, then
   (a) 0 is an eigenvalue of A. (b) $A^4 = 0$ (c) A is not diagonalizable. (d) None of these.

9. Let A be an $n \times n$ non-zero nilpotent matrix over $\mathbb{R}$. Then
   (a) A is diagonalizable. (b) A is diagonalizable if n is odd.
   (c) A is not diagonalizable. (d) None of the above.

10. Let $A = \begin{pmatrix} \alpha & -3 \\ 3 & 0 \end{pmatrix}$, $\alpha \in \mathbb{R}$ is a parameter. Then
    (a) A is not diagonalizable for any $\alpha \in \mathbb{R}$. (b) A is diagonalizable $\forall \alpha \mathbb{R}$.
    (c) A is not diagonalizable if $-6 \leq \alpha \leq 6$. (d) A is diagonalizable if $-6 < \alpha < 6$.

11. Let A and B be $n \times n$ matrices over $\mathbb{R}$ such that $AB = A - B$. If B is a diagonalizable matrix with only one eigenvalue 2, then,
   (a) 2 is also an eigenvalue of A. (b) A is diagonalizable and -2 is the only eigenvalue of A.
   (c) A may not be diagonalizable. (d) None of these.

12. The matrix $A = \begin{pmatrix} 1 & 7 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{pmatrix}$

   (a) Not diagonizable. (b) is similar to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

   (c) is similar to $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. (d) None of the above.

13. Let A, B, C be 3×3 non-diagonal matrices over $\mathbb{R}$ such that
   $A^2 = A, B^2 = -I, (C - 3I)^2 = 0$. Then
   (a) A, B, C are all diagonalizable over $\mathbb{R}$. (b) A, C are all diagonalizable over R.
   (c) Only A is diagonalizable over $\mathbb{R}$. (d) None of the above

14. Let $A \in M_3(\mathbb{R})$ such that $AB = BA$ for all $B \in M_3(\mathbb{R})$. Then
   (a) A has distinct eigenvalues and is diagonalizable.
   (b) A is not diagonalizable.
   (c) A does not have distinct eigenvalues but is diagonalizable.
   (d) None of the above.

15. If $A, B, C, D \in M_2(\mathbb{R})$ such that A, B, C, D are non-zero and not diagonal. If $A^2 = I, B^2 = B, C^2 = 0, C \neq 0$ and every eigenvalue of D is 2, then
   (a) A, B, C, D are all diagonalizable. (b) B, C, D are diagonalizable.
   (c) A, B are diagonalizable. (d) Only D is diagonalizable.

16. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then

   (a) Both A, B are diagonalizable, A is also orthogonally diagonalizable.

   (b) Both A, B are orthogonally diagonalizable.

   (c) Both A, B are diagonalizable, B is also orthogonally diagonalizable.

   (d) Both A, B are diagonalizable, but both A, B are not orthogonally diagonalizable.

Practical 2.5. Descriptive Questions

1. In each following matrices A,
   (a) Find eigen values of A, geometric and algebraic multiplicity of each eigen values.
   (b) Determine whether A is diagonalizable. In case, the given matrix is diagonalizable find a non-singular matrix P so that $P^{-1}AP$ is a diagonal matrix.

(i) $\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}$ (ii) $\begin{pmatrix}1 & 2\\ 3 & 6\end{pmatrix}$ (iii) $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

(iv) $\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$ (v) $\begin{pmatrix}1 & 1 & 0\\ 0 & 2 & 2\\ 0 & 0 & 4\end{pmatrix}$ (vi) $\begin{pmatrix}3 & 1 & 6\\ 2 & 1 & 0\\ -1 & 0 & -3\end{pmatrix}$

(vii) $\begin{pmatrix}3 & 0 & 6\\ 0 & -3 & 0\\ 5 & 0 & 2\end{pmatrix}$ (viii) $\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 3 & 0 & 1\end{pmatrix}$ (ix) $\begin{pmatrix}3 & 0 & -2\\ -7 & 0 & 4\\ 4 & 0 & -3\end{pmatrix}$

2. For the following find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix.

   (a) $A = \begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}$.Hence find A^{1000}.

   (b) $A = \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$.Hence find A^{10}.

   (c) $A = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$.Hence find A^{100}.

   (d) $A = \begin{pmatrix}1 & -2 & 8\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$.Hence find $A^{1000}, A^{-1000}, A^{2002}, A^{2003}$.

3. Determine constants a, b, c so that the matrix $A = \begin{pmatrix}1 & a & b\\ 0 & 2 & c\\ 0 & 0 & 1\end{pmatrix}$ is diagonalizable.

4. Characterize the diagonalizable 2×2 matrices A such that $A^2 - 3A + 2I = 0$ in terms of their eigenvalues.

5. Show that $A = \begin{pmatrix}a & b\\ 0 & d\end{pmatrix}$, $a, b, d \in \mathbb{R}$ is diagonalizable if and only if $b = 0$ or $a \neq d$.

6. (i) Is the matrix $\begin{pmatrix}1 & 6\\ 2 & 0\end{pmatrix}$ is similar to $\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}$.

   (ii) Is the matrix $\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}$ is similar to $\begin{pmatrix}3 & 0\\ 1 & 2\end{pmatrix}$.

7. Let $A = \begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}$. Find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix and hence find A^{100}.

8. Find a 3×3 matrix A which has eigenvalues $0, 1and - 1$ with corresponding eigenvectors $(0, 1. - 1)^t, (1; -1; 1)^t, (0, 1, 1)^t$ respectively.

9. Find the eigenvalues and eigenvectors of 13×13 matrix $A = \begin{pmatrix}0 & \cdots & 0 & 1\\ 0 & \cdots & 0 & 1\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 0 & 1\\ 1 & \cdots & 1 & 1\end{pmatrix}$ and show it is diagonalizable.

10. Let $A_{n\times n}$ real matrix such that $A^2 = A$. Prove that A is diagonalizable.

Practical no 2.6 Orthogonal Diagonalization and Quadratic Form

1. If $v = [1, 0, 1]$ is a row vector then,
   (a) $v^t v$ is not orthogonally diagonalizable.
   (b) $vv^t v$ is orthogonally diagonalizable.

   (c) $v^t v$ is not diagonalizable.
   (d) None of the above.

2. Let A be an $m \times n$ matrix over $\mathbb{R}$. Then
   (a) AA^t is not orthogonally diagonalizable.
   (b) $I_m + AA^t$ is not orthogonally diagonalizable.
   (c) AA^t and $A^t A$ are orthogonally diagonalizable. (d) None of the above.

3. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. If $P^t AP = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, then $P =$
   (a) $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ (c) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ (d) None of the above.

4. Let $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, $a \in \mathbb{R}$. Then
   (a) A is not diagonalizable for any $a \in \mathbb{R}$.
   (b) A is diagonalizable but not orthogonally diagonalizable.
   (c) A is orthogonally diagonalizable if and only if $a = 0$. (d) None of the above.

5. The equation $2x^2 - 4xy - y^2 - 4x + 10y - 13 = 0$ after rotation and translation can be reduced to
   (a) an ellipse (b) a hyperbola (c) a parabola (d) a pair of straight lines.

6. The conic $x^2 + 2xy + y^2 = 1$ reduces to the standard form after rotation through a angle
   (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{6}$

7. The quadratic form $Q(x) = x_1^2 + 4x_1x_2 + x_2^2$ has
   (a) rank = 1, signature = 1. (b) rank = 2, signature = 0.
   (c) rank = 2, signature = 2. (d) None of the above.

8. Let A be a 4×4 real symmetric matrix. Then there exists a 4×4 real symmetric matrix B such that
   (a) $B^2 = A$ (b) $B^3 = A$ (c) $B^4 = A$ (d) None of these

9. The matrix $\begin{pmatrix} 1 & 2 \\ 2 & k \end{pmatrix}$ is positive definite if
   (a) $k > 4$ (b) $-2 < k < 2$ (c)$|k| > 2$ (d) None of these.

10. $ax^2 + bxy + cy^2 = d$ where a, b, c are not all zero and $d > 0$ represents
    (a) ellipse if $b^2 - 4ac > 0$ and hyperbola if $b^2 - 4ac < 0$.
    (b) ellipse if $b^2 - 4ac < 0$ and hyperbola if $b^2 - 4ac > 0$.
    (c) is a circle if $b = 0$ and $a = c$ else it is a hyperbola.
    (d) None of these.

11. The conic $x^2 + 10x + 7y = -32$ represents
(a) a hyperbola (b) an ellipse. (c) a parabola (d) a pair of straight lines.

12. For the quadratic from $Q(x) = 2x_1^2 + 2x_2^2 - 2x_1x_2$
(a) rank = 2, signature = 1 (b) rank = 1, signature = 1
(c) rank = 2, signature = 0 (d) rank = 2, signature = 2

13. For the quadratic from $Q(x) = -3x_1^2 + 5x_2^2 + 2x_1x_2$,
(a) rank = 2, signature = 0 (b) rank = 2, signature = 1
(c) rank = 2, signature = 2 (d) rank = 1, signature = 1

14. The symmetric matrix associated to the quadratic from $5(x_1 - x_2)^2$ is,
(a) positive definite (b) positive semi definite (b) indefinite (d) negative definite.

15. The quadratic form $Q(x) = 2x_1^2 - 4x_1x_2 - x_2^2$ after rotation can be reduced to standard form
(a) $3y_1^2 - 2y_2^2$ or $2y_1^2 + 3y_2^2$ (b) $3y_1^2 + 2y_2^2$ (c) $-3y_1^2 + 2y_2^2$ (d) $2y_1^2 - 4y_2^2$

16. The equation $x^2 + y^2 + z^2 - 2x + 4y - 6z = 11$ represents
(a) None of the below (b) a hyperboloid of one sheet
(c) a hyperboloid of two sheet (d) a sphere.

17. The conic $3x^2 - 4xy = 2$ represents
(a) an ellipse (b) a hyperbola (c) a parabola (d) a pair of straight lines.

18. Let $Q(X) = X^tAX$, where $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $X = (x_1, x_2, x_3, x_4)^t$. Then by orthogonal change of variable, $Q(X)$ can be reduced to
(a) $y_1y_2 + y_3^2$ (b) $y_1y_2 + y_2^2 + y_3^2$
(c) $y_1^2 + y_2^2 + y_3^2 - y_4^2$ (d) $y_2^2 + y_2^2 - y_3y_4$

19. If $A_{n\times n}$ be real matrix then which of the following is true-
(a) A has at least one eigen value. (b) $\forall X, Y \in \mathbb{R}$, $\langle AX, AY\rangle > 0$
(c) Each eigen value of $A^tA \geq 0$ (d) A^tA has n eigen values.

Practical 2.6 Descriptive Questions

1. Find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix, in each of the following examples. $A =$
(a) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 0 & 7 \\ 0 & 5 & 0 \\ 7 & 0 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{pmatrix}$

2. In the following examples, make an orthogonal change of variables $X = PY$ to reduce the given quadratic form $Q(X)$ to standard form $\sum_{i=1}^{n} \lambda_i y_i^2$. Also, state rank and signature of $Q(X)$.

   (a) $Q(x_1, x_2) = 11x_1^2 + 6x_1x_2 + 19x_2^2$.

(b) $Q(x_1,x_2) = x_1^2 + 4x_1x_2 + x_2^2$.

(c) $Q(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2 - 4(x_1x_2 + x_2x_3 + x_1x_3)$.

(d) $Q(x_1,x_2,x_3) = 7x_1^2 + x_2^2 + x_3^2 + 8x_1x_2 + 8x_1x_3 - 16x_2x_3$.

(e) $Q(x_1,x_2,x_3) = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_1x_3 - x_2x_3)$.

(f) $Q(x_1,x_2,x_3) = 5x_1^2 + 8x_2^2 + 5x_3^2 - 4(x_1x_2 + 2x_1x_3 + x_2x_3)$.

(g) $Q(x_1,x_2,x_3) = x_1^2x_3^2 - 4(x_1x_2 + 4x_2x_3)$.

(h) $Q(x_1,x_2,x_3) = x_1^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$.

3. Find rank and signature of the following

$$(1)\ \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix} \quad (2)\ \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad (3)\ \begin{pmatrix} 0 & 5 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

4. In each of the following examples, find value of k, for which the symmetric matrix associated to the quadratic form is positive definite.

(a) $x_1^2 + kx_2^2 - 4x_1x_2$.

(b) $5x_1^2 + x_2^2 + kx_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$.

(c) $3x_1^2 + x_2^2 + 2x^3 + 2x_1x_3 + 2kx_2x_3$.

5. In each of the following examples, a translation in $\mathbb{R}^2$ puts the conic in standard form. Reduce the conic to standard form and identify it.
(a) $9x^2 + 4y^2 - 36x - 24y + 36 = 0$.
(b) $x^2 + 10x + 7y = -32$.
(c) $y^2 - 8x - 14y + 49 = 0$.
(d) $x^2 - 16y^2 + 8x + 128y = 256$.
(f) $x^2 + y^2 + 6x - 10y + 18 = 0$

6. In each of the following examples, a rotation of coordinate axes reduces the conic to standard form. Identify the conic and give its equation in the standard form in the rotated system.
(a) $2x^2 - 4xy - y^2 + 8 = 0$. (b)$x^2 + 2xy + y^2 - 2 = 0$. (c) $5x^2 + 4xy + 5y^2 = 9$.

7. In each of the following examples a rotation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ reduces the quadric to standard form. Name the quadric and give its equation in $x'y'z'$ system.
(a) $2x^2 + 3y^2 + 23z^2 + 72xz + 150 = 0$ (b) $4x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 4yz - 5 = 0$.

8. Reduce the following quadratic forms to standard form:
(a) $4xz + 4y^2 + 8y + 8$
(b) $9x^2 - 6xy + 6y^2 + 2\sqrt{5}x + 12y + 16z$.
(c) $x^2 + 4y^2 + 4z^2 + 4xy - 4xz - 8yz + 2x + 8y + 7$

9. let $A = \begin{bmatrix} 0 & 5 & 1 & 0 \\ 5 & 0 & 5 & 0 \\ 1 & 5 & 0 & 5 \\ 0 & 0 & 5 & 0 \end{bmatrix}$. Let $Q(X) = X^tAX$, find rank and signature of $Q(X)$.

Practical 2.7
Unit 1

Q 1) Let W be a subspace of a real vector space V. For $v_1, v_2 \in V$, show that

(i) $(v_1 + W) = (v_2 + W)$ if and only if $v_1 - v_2 \in W$.

(ii) either $(v_1 + W) \cap (v_2 + W) = \{0\}$ or $(v_1 + W) = (v_2 + W)$

Q 2) Let W be a subspace of a real vector space V and $V/W = \{v + W \ / \ v \in V\}$. Show that addition defined by $(v_1+W)+(v_2+W) = (v_1+v_2)+W$ and scalar multiplication defined by $\alpha.(v + W) = \alpha.v + W$ are well defined in V/W.

Q 3) Let W be a subspace of a finite dimensional vector space V. Show that $\dim V/W = \dim V - \dim W$.

Q 4) Let $P_n[\mathbb{R}]$ denote the space of polynomials with real coefficients of degree $\leq n$ along with zero polynomial. Consider the linear transformation $D : P_n[\mathbb{R}] \to P_{n-1}[\mathbb{R}]$ defined by $D(f) = \frac{df}{dx}$ and $T : P_n[\mathbb{R}] \to P_{n+1}[\mathbb{R}]$ defined by $T(f) = xf$. If $A = DT - TD : P_n[\mathbb{R}] \to P_n[\mathbb{R}]$, find $KerA$, and $dim(A/KerA)$

Q 5) State and prove the 'First Isomorphism Theorem of vector space' (Fundamental theorem of vector space homomorphism).

Q 6) Show that any orthogonal linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is either a rotation about origin or a reflection about a line passing through origin.

Q 7) Let V be a finite dimensional inner product vector space and $T : V \to V$ be a linear transformation. Prove that the following statements are equivalent.

(i) T is orthogonal.

(ii) $\|T(X)\| = \|X\|$ for all $X \in V$.

(iii) If $\{e_i\}_{i=1}^n$ is an orthonormal basis of V, then $\{T(e_i)\}_{i=1}^n$ is also an orthonormal basis of V.

Q 8) Let V be a finite dimensional inner product vector space. If $f : V \to V$ is a function such that (i) $f(0) = 0$ (ii) $\|f(X) - f(Y)\| = \|X - Y\|$, $\forall\ X, Y \in V$, then show that f is an orthogonal linear transformation.

Q 9) Let V be a finite dimensional inner product vector space and $f : V \to V$ be an isometry, then show that there exists unique $x_0 \in V$ and an unique orthogonal linear transformation $T : V \to V$ such that $f = L_{x_0} \circ T$ where $L_{x_0} : V \to V$ is a translation map defined as $L_{x_0}(X) = X + X_0$.

Q 10) Let V be an n dimensional inner product space and W be a subspace of V of dimension $n - 1$. Let u be a unit vector orthogonal to W. Show that $T : V \to V$ defined by $T(x) = x - 2\langle x, u\rangle u$ is an orthogonal linear transformation such that $T(w) = w$, $\forall\ w \in W$ and $T(u) = -u$.

Q 11) Let A be a $n \times n$ real matrix. Show that

(i) $\det xI_n - A$ is a monic polynomial of degree n in $'x'$.

(ii) Coefficient of x^{n-1} in the polynomial is = - tr A.

(iii) Constant term of the polynomial is $= (-1)^n$ det A.

Q 12) State and prove the Cayley Hamilton Theorem.

Q 13) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $\langle u, v\rangle = 0 \quad \Rightarrow \langle T(u), T(v)\rangle = 0$ for each $u, v \in \mathbb{R}^2$, show that $T = aS$, where $S : \mathbb{R}^2 \to \mathbb{R}^2$ is an orthogonal transformation.

Unit 2

Q 1) Let V be a vector space of finite dimension 'n'and $T : V \to V$ be a linear transformation. Show that following statements are equivalent.

(i) $\lambda \in \mathbb{R}$ is a an eigen value of T.

(ii) $\lambda I_n - T$ is not injective. ($I : V \to V$ is the identity map.)

(iii) λ is an eigen value of a matrix A,where A is a matrix associated with T with respect to any basis of V.

(iv) λ is a root of the characteristic polynomial of A.

(v) The system of homogeneous linear equations $[\lambda I_n - A]X = 0$ has non-zero solution $X \in \mathbb{R}^n$.

Q 2) If λ is an eigen value of a real $n \times n$ matrix A, then

(i) λ is an eigen value of A^t.

(ii) λ^k is an eigen value of A^k for $k \in \mathbb{N}$. Hence $f(\lambda)$ is an eigen value of $f(A)$, for a polynomial $f(x)$ over $\mathbb{R}$.

(iii) If A is invertible, then λ^{-1} is an eigen value of A^{-1}.

Q 3) For $n \times n$ real matrix A, prove that -

(i) Characteristic polynomial of A = Characteristic polynomial of A^t.

(ii) Characteristic polynomial of B = characteristic polynomial of A for any matrix B similar to A.

(iii) For any real matrix $C_{n\times n}$, Characteristic polynomial of AC = characteristic polynomial of CA.

Q 4) If A is an $n \times n$ real matrix, and $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigen value of A with $X_1, X_2, \cdots, X_k$ as corresponding eigenvectors, then show that $X_1, X_2, \cdots, X_k$ are linearly independent.

Or

If $T : V \to V$ is a linear transformation where V is an vector space of dimension n and $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigen value of T with $X_1, X_2, \cdots, X_k$ as corresponding eigenvectors, then show that $X_1, X_2, \cdots, X_k$ are linearly independent in V.

Q 5) Let A be a $n \times n$ matrix having 'n' eigen values, then prove that A is similar to an upper triangular matrix.

Q 6) Show that minimal polynomial of a real matrix $A_{n\times n}$ divides every polynomial which annihilates A. Hence the minimal polynomial of A divides the characteristic polynomial of A.

Q 7) α is a root of the minimal polynomial of matrix A if and only if α is a characteristic root of A.

Q 8) Similar matrices have same minimal polynomials.

Q 9) Define invariant subspace. Let V be a finite dimension vector space and $T : V \to V$ be a linear transformation. Show that

(a) $ker\, T, Im\, T$ are invariant under T.

(b) eigen space of T is invariant under T.

(c) If V is an inner product space, T is symmetric (i.e. $\langle TX, Y\rangle = \langle X, TY\rangle\ \forall X, Y \in V$) and W is invariant under T, then $W^{\perp}$ is also invariant under T.

Unit 3

1. Define a diagonalizable matrix. If A is an $n \times n$ real matrix, and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct eigen values of A, show that A is diagonalizable.

2. Define Algebraic and Geometric multiplicity of an eigen value of a square matrix. Show that the algebraic multiplicity of an eigen value does not exceed its geometric multiplicity.

3. For $n \times n$ matrix A, show that following are equivalent-
   (a) A is diagonalizable.
   (b) $\mathbb{R}^n$ has basis consists of eigen vectors of A.
   (c) There are n eigen values to A and algebraic multiplicity of each eigen value coincides with its geometric multiplicity.
   (d) sum of dimensions of eigen spaces of A is n.

4. Let V be an n dimensional vector space and $T : V \to V$ be linear transformation. When do we say that T is diagonalizable? Show that T is diagonalizable if and only if V has a basis consists of eigen vectors of T.

5. Let A be real symmetric matrix of order n. Show that eigen values of A are real. Also show that if $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigen values of A and $X_1, X_2, \cdots X_k$ are corresponding eigen vectors then $\{X_1, X_2, \cdots, X_k\}$ form an orthogonal set.

6. Define an orthogonally diagonalizable matrix. Show that every real symmetric matrix is orthogonally diagonalizable.

7. Define a quadratic form in n variables. Define symmetric associated to it. What is the standard (normal or canonical) form of quadratic form? Show that every quadratic form $Q(x_1, x_2, \cdots, x_n)$ can be reduced to standard form by suitable change of variables.

or

Show that every quadratic form $Q(x_1, x_2, \cdots x_n)$ over $\mathbb{R}$ can be to reduced standard form $\sum_{i=1}^{n} \lambda_i y_i^2$ by an orthogonal change of variables $X = PY$,$X = (x_1, x_2 \cdots, x_n)^t$, $y = (y_1, y_2, \cdots y_n)^t$ and P is an $n \times n$ orthogonal matrix.

8. Define a positive definite quadratic form $Q(x_1, x_2, \cdots, x_n)$. Show that quadratic form Q is positive definite if and only if Rank Q = sign $Q = n$.

or

Show that a quadratic form Q is positive definite if and only if all eigen values of associated symmetric matrix are positive.

or

Let A be an $n \times n$ real symmetric matrix. Then show that the following statements are equivalent.
(i) $\langle AX, X\rangle > 0$ for all non-zero $X \in \mathbb{R}^n$. (ii) Each eigenvalue of A is positive.

9. Define positive definite symmetric matrix. Show that a symmetric matrix is positive definite if and only if all leading principal minors of A are positive.

10. Consider the equation $f(x, y) = ax^2 + 2fxy + by^2 + cx + dy + e = 0$. Show that by applying rotation and translation the equation $f(x, y)$ reduces to $f(X, Y) = \lambda_1(X - h)^2 + \lambda_2(Y - k)^2 + L$; $h, k, L \in \mathbb{R}$.

Practical 3.1 : Examples of Metric Spaces, Normed Linear Spaces.
Objective Questions 3.1

(1) Consider the following maps $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$.
(i) $d(x, y) = |x - 2y|$ (ii) $d(x, y) = |x^2 - y^2|$
(iii) $d(x, y) = |x - y|^2$ (iv) $d(x, y) = |x - y|^{\frac{1}{2}}$
(a) (iii) and (iv) are metrics on $\mathbb{R}$
(b) Only (iv) is a metric on $\mathbb{R}$
(c) (ii) and (iii) are metrics on $\mathbb{R}$
(d) (ii), (iii) and (iv) are metrics on $\mathbb{R}$

(2) Let d_1 and d_2 be metrics on a non-empty set X. Then
(a) $d_1^2 + d_2^2, ad_1$ where $a > 0$ are metrics on X, where $(d_1^2 + d_2^2)(x, y) = \Big(d_1(x, y)\Big)^2 + \Big(d_2(x, y)\Big)^2$ and $(ad_1)(x, y) = a\Big(d_1(x, y)\Big)$
(b) $\sqrt{d_1} + \sqrt{d_2}, ad_1$ where $a > 0$ are metrics on X, where $(\sqrt{d_1} + \sqrt{d_2})(x, y) = \sqrt{d_1(x, y)} + \sqrt{d_2(x, y)}$ and $(ad_1)(x, y) = a\Big(d_1(x, y)\Big)$
(c) $ad_1 + bd_2$ where $a, b \in \mathbb{R}$ is a metric on X, where $(ad_1 + bd_2)(x, y) = ad_1(x, y) + bd_2(x, y)$
(d) None of the above

(3) Consider the discrete metric d_1 defined on a non-empty set X by $d_1(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$.
Then for $x, y, z \in X$,
(a) $d_1(x, z) < d_1(x, y) + d_1(y, z)$
(b) $d_1(x, z) < d_1(x, y) + d_1(y, z)$ if and only if x, y, z are distinct.
(c) $d_1(x, z) = d_1(x, y) + d_1(y, z)$ if and only if $x = y = z$
(d) None of the above

(4) Let d_1 and d_2 be metrics on a non-empty set X. For $x, y \in X$, let $d(x, y) = \min\,\{d_1(x, y), d_2(x, y)\}$ and $d'(x, y) = \max\,\{d_1(x, y), d_2(x, y)\}$. Then
(a) Both d, d' are metrics on X. (b) d is a metirc on X, d' is not.
(c) d' is a metirc on X, d is not. (d) None of the above.

(5) Let (X, d_1) and (Y, d_2) be metric spaces. $d, d', d'' : (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$ are defined as follows:
(i) $d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$
(ii) $d'((x_1, y_1), (x_2, y_2)) = [(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2]^{\frac{1}{2}}$
(iii) $d''((x_1, y_1), (x_2, y_2)) = [(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2]$
(a) d, d', d'' are all metrics on $X \times Y$ (b) d, d' are metrics on $X \times Y$
(c) d', d'' are metrics on $X \times Y$ (d) None of the above.

(6) Let $(X, \|\ \|)$ be a normed linear space and $x, y, z \in X$. If d is the metric induced by the norm then

(a) $d(x+z, y+z) \geq d(x, y)$ and the strict inequality may hold.
(b) $d(x+z, y+z) \geq d(x, y) + d(y, z)$ and the strict inequality may hold.
(c) $d(x+z, y+z) = d(x, y)$.
(d) None of the above

(7) Consider the norms $\| \|_1, \| \|_2$ and $\| \|_\infty$ on $\mathbb{R}^2$, $\|x\|_1 = |x_1| + |x_2|$, $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$, $\|x\|_\infty = \max\{|x_1|, |x_2|\}$. Then
(a) $2\|x\|_\infty \leq \|x\|_2 \leq 2\|x\|_1$ (b) $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$
(c) $2\|x\|_\infty \leq \|x\|_1 \leq 2\|x\|_2$ (d) None of the above

(8) Let $X = C[0,1]$ and consider the norms $\| \|_1, \| \|_\infty$ on X, where $\|f\|_1 = \int_0^1 |f(t)|\, dt$, $\|f\|_\infty = \sup\{|f(t)|, t \in [0,1]\}$. Then for $f(t) = t, g(t) = t^2 \in X$, if d_1 and d_∞ are metric induced by $\| \|_1$, and $\| \|_\infty$ then
(a) $d_1(f,g) = \frac{1}{2}, d_\infty(f,g) = \frac{1}{3}$ (b) $d_1(f,g) = \frac{1}{6}, d_\infty(f,g) = \frac{1}{4}$
(c) $d_1(f,g) = \frac{1}{3}, d_\infty(f,g) = \frac{1}{2}$ (d) None of the above.

(9) Consider the normed linear space $(l^2, \| \|_2)$ where $l^2 = \{(x_n) : (x_n)$ is a sequence over $\mathbb{R}$, such that $\sum_{n=1}^{\infty} x_n^2 < \infty\}$ and for $x = (x_1, x_2, \ldots, x_n, \ldots)$, $\|x\|_2 = \sqrt{\sum_{n=1}^{\infty} x_n^2}$. Let $e_1 = (1,0,0,\ldots), e_2 = (0,1,0,0,\ldots)$. Then for the metric d_2 induced by $\|\|_2$,
(a) $d_2(e_1+e_2, e_1-e_2) = \sqrt{2}$ (b) $d_2(e_1+e_2, e_1-e_2) = 2$
(c) $d_2(e_1+e_2, e_1-e_2) = \frac{1}{\sqrt{2}}$ (d) None of the above.

(10) Let X be the set of all real sequences $x = (x_n)$. Consider the metric d defined by

$$\begin{aligned} d(x,y) &= 0 && \text{if } x = y \\ &= \frac{1}{\min\{i : x_i \neq y_i\}} && \text{if } x \neq y \end{aligned}$$

where $x = (x_n), y = (y_n) \in X$. Then for distinct sequences $x, y, z \in X$
(a) $d(x,z) \leq d(x,y) + d(y,z)$ and the equality may hold.
(b) $d(x,z) \leq \max\{d(x,y), d(y,z)\}$
(c) $d(x,z) \geq \max\{d(x,y), d(y,z)\}$
(d) None of the above.

(11) Let $(X, \| \|)$ be a normed linear space and d be the metric induced by $\| \|$. Then for $x, y, z \in X, d(x,z) = d(x,y) + d(y,z)$ if and only if
(a) $y = z$
(b) y lies on the segment joining x and z and between them.
(c) z lies on the segment joining x and y and between them.
(d) None of the above.

(12) Let X be a normed linear space and $x, y \in X$. Then
(a) $\|x-y\| \leq |\ \|x\| - \|y\|\ |$ (b) $\|x-y\| = |\ \|x\| - \|y\|\ |$
(c) $\|x-y\| \geq |\ \|x\| - \|y\|\ |$ (d) None of the above.

(13) Let $X = M_2(\mathbb{R})$. Consider the following maps from $X \longrightarrow \mathbb{R}$.
(i) $\|A\| = |\det A|$
(ii) $\|A\| = \sum_{1\leq i,j\leq 2} |a_{ij}|$ where $A = (a_{ij})$
(iii) $\|A\| = \max_{1\leq i,j\leq 2} |a_{ij}|$ where $A = (a_{ij})$
Then
(a) (i), (ii), (iii) are all norms on X.
(b) (ii) and (iii) are norms on X.
(c) (i) and (ii) are norms on X.
(d) None of the above.

Topology of Metric Spaces: Practical 3.1
Examples of Metric Spaces, Normed Linear Spaces
Descriptive Questions 3.1

(1) Let d_1 and d_2 be metrics on a non-empty set X. Check if the following are metrics on X. Justify your answer.

(i) d, where $d(x,y) = \max\{d_1(x,y), d_2(x,y)\}$ for $x, y \in X$
(ii) d, where $d(x,y) = \min\{d_1(x,y), d_2(x,y)\}$ for $x, y \in X$
(iii) d, where $d(x,y) = 2d_1(x,y) + 3d_2(x,y)$ for $x, y \in X$
(iv) d, where $d(x,y) = (d_1(x,y))^2 + (d_2(x,y))^2$ for $x, y \in X$
(v) d, where $d(x,y) = \max\{1, d_1(x,y), d_2(x,y)\}$ for $x, y \in X$

(2) Let (X, d) be a metric space. Show that the following are metrics on X.

(i) d_1 where $d(x,y) = \sqrt{d(x,y)}$
(ii) d, where $d(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$

(3) Show that d is a metric on $\mathbb{R}$, where $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ |x| + |y| & \text{if } x \neq y, x, y \in \mathbb{R} \end{cases}$

(4) Let $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n\}$. Show that $\|\ \|_1, \|\ \|_2$, and $\|\ \|_\infty$ are norms on $\mathbb{R}^n$ where for $x = (x_1, x_2, \ldots, x_n)$, $\|x\|_1 = \sum_{i=1}^{i=n} |x_i|$, $\|x\|_2 = \sqrt{\sum_{i=1}^{i=n} x_i^2}$ and $\|x\|_\infty = \max\{|x_1| : 1 \leq i \leq n\}$. Further, show that $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ and $\|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty$ for $x \in \mathbb{R}^n$

(5) Let $l^2 = \{(x_n) : (x_n)$ is a sequence of real numbers such that $\sum_{n=1}^{\infty} x_n^2 < \infty\}$. If $\|x\|_2 = \left(\sum_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}}$ for $x = (x_1, x_2, \ldots, x_n, \ldots) \in l^2$, then show theat $(l^2, \|\ \|_2)$ is a normed linear space.

(6) Let $X = C[0,1]$ and show that $\|\ \|_1 : X \longrightarrow \mathbb{R}$ and $\|\ \|_\infty : X \longrightarrow \mathbb{R}$ defined by,
$\|f\|_1 = \int_0^1 |f(t)|\, dt, \quad \|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}$ are norms on X

(7) Let $X = C[0,1]$ and consider the norms $\|\ \|_1$ and $\|\ \|_\infty$ defined by,
$\|f\|_1 = \int_0^1 |f(t)|\, dt, \quad \|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}$
Then for $f = t, g = t^2, h = t^3, t \in [0,1]$, find $d_1(f,g), d_\infty(f,g), d_1(f,h), d_\infty(f,h)$ where d_1 and d_∞ are metrics induced by the norms $\|\ \|_1$ and $\|\ \|_\infty$ respectively.

(8) Let X be the set of real sequences

(i) Show that $d : X \times X \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} d(x,y) &= 0 && \text{if } x = y \\ &= \frac{1}{\min\{i : x_i \neq y_i\}} && \text{if } x \neq y \end{aligned}$$

where $x = (x_n), y = (y_n) \in X$ is a metric on X.

(ii) Show that $d : X \times X \longrightarrow \mathbb{R}$ defined by

$$d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}$$

where $x = (x_n), y = (y_n) \in X$ is a metric on X.

(iii) Let $X = \{(x_n) : (x_n)$ is a sequence of real numbers, $x_n \longrightarrow 0\}$. Show that $\|\ \| : X \longrightarrow \mathbb{R}$ defined by $\|x\| = \sup\{|x_n| : n \in \mathbb{N}\}$ for $x = (x_n)$ is a norm on X.

(9) Let $\|\ \|_2$ be the Euclidean norm on $\mathbb{R}^2$. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$\begin{aligned} d(x,y) &= \|x\|_2 + \|y\|_2 && \text{if } x \neq y \\ &= 0 && \text{if } x = y \end{aligned}$$

for $x, y \in \mathbb{R}^2$. Show that d is a metric on $\mathbb{R}^2$

(10) Show that d is a metric on $\mathbb{N}$ where for $m, n \in \mathbb{N}$,

$$\begin{aligned} d(m,n) &= 0 && \text{if } m = n \\ &= 1 + \frac{1}{m+n} && \text{if } m \neq n \end{aligned}$$

(11) Show that $\| \ \|$ is a norm on X, where $X = M_2(\mathbb{R})$ and $\|A\| = \max_{1\leq i,j\leq 2}|a_{ij}|$ for $A = (a_{ij})$

(12) Show that $\| \ \|_1$ is a norm on l^1 where $l^1 = \left\{(x_n) : x_n \in \mathbb{R}, \sum_{n=1}^{\infty}|x_n| < \infty\right\}$ and $\|x\|_1 = \sum_{n=1}^{\infty}|x_n|$ for $x = (x_n)$

(13) Show that $\mathbb{C}$ (set of complex numbers) is a normed linear space where norm is the absolute value of a complex number.

Topology of Metric Spaces Practical 3.2
Sketching of Open Balls in $\mathbb{R}^2$, Open and Closed sets, Equivalent metric spaces
Objective Questions 3.2

. (Revised Syllabus 2018-19)

(1) In a metric space (X, d)
  (a) an arbitrary intersection of open sets is an open sets.
  (b) an arbitrary intersection of open balls is an open ball.
  (c) an intersection of finitely many open balls is an open ball.
  (d) None of the above.

(2) Let (X, d) be a metric space and $x, y \in X, r, s > 0$. If $B(x, r) = B(y, s)$, then
  (a) $x = y$ and $r = s$ (b) $x = y$ but r may not be equal to s
  (c) $r = s$ (d) None of the above

(3) Let (X, d) be a metric space and $x, y \in X, 0 < r < s$. Then
  (a) $B(x, r) \subseteq B(x, s)$ and the equality may occur. (b) $B(x, r) \subsetneq B(x, s)$,
  (c) $B(x, r) = B(x, s)$ if $r \geq 1$ (d) None of the above.

(4) Let (X, d) be a metric space in which the only open subsets are $\emptyset$ and X. Then
  (a) d is a discrete metric on X.
  (b) For $x, y \in X, d(x, y) \geq 1$ if $x \neq y$
  (c) X is a singleton set.
  (d) None of the above.

(5) Let G be a non-empty bounded open set in $\mathbb{R}^2$ with Euclidean metric. Then G is of the type
  (a) $(a, b) \times (c, d)$, where $a, b, c, d \in \mathbb{R}, a < b, c < d$.
  (b) $I \times J$, where I and J are union of finitely many bounded open intervals in $\mathbb{R}$
  (c) $G_1 \times G_2$, where G_1 and G_2 are bounded open subsets of $\mathbb{R}$.
  (d) None of the above.

(6) Consider the normed linear space $(\mathbb{R}^2, \| \|_1)$ where for $x = (x_1, x_2) \in \mathbb{R}^2, \|x\|_1 = |x_1| + |x_2|$. If $B_1((0, 0), 1)$ is an open ball with center $(0, 0)$ and radius 1, then
  (a) $B_1((0, 0), 1)$ is a square with sides of length $\sqrt{2}$ which are parallel to coordinate axes.
  (b) $B_1((0, 0), 1)$ is a square with sides of length $\sqrt{2}$ and diagonals are parallel to coordinate axes.
  (c) $B_1((0, 0), 1)$ is a square with sides of length 2 which are parallel to coordinate axes.
  (d) None of the above.

(7) Let (X, d) be a metric space and $x, y \in X$. Let $d(x, y) = s > 0$. Then $B(x, r) \cap B(y, r) = \emptyset$, if
  (a) $r \geq \frac{s}{2}$ (b) $0 < r \leq \frac{s}{2}$ (c) $r \geq 2s$ (d) None of the above

(8) Consider the normed linear spaces $(\mathbb{R}^2, \| \ \|_1), (\mathbb{R}^2, \| \ \|_2)$ and $(\mathbb{R}^2, \| \ \|_\infty)$ where for $x = (x_1, x_2) \in \mathbb{R}^2$ $\|x\|_1 = |x_1| + |x_2|, \|x\|_2 = \sqrt{x_1^2 + x_2^2}, \|x\|_\infty = \max\{|x_1|, |x_2|\}$
If $B_1((0,0),1), B_2((0,0),1)$ and $B_\infty((0,0),1)$ denote open balls in $(\mathbb{R}^2, \| \ \|_1), (\mathbb{R}^2, \| \ \|_2)$ and $(\mathbb{R}^2, \| \ \|_\infty)$ respectively. Then
(a) $B_1((0,0),1) \subsetneq B_2((0,0),1) \subsetneq B_\infty((0,0),1)$
(b) $B_1((0,0),1) = B_2((0,0),1) = B_\infty((0,0),1)$
(c) $B_\infty((0,0),1) \subsetneq B_2((0,0),1) \subsetneq B_1((0,0),1)$
(d) None of the above.

(9) Let (X, d) be a metric space aned d_1 be the metric on X defined by $d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$ for $x, y \in X$
(a) Every open ball in (X, d_1) is an open ball in (X, d) and viceversa.
(b) Every open ball in (X, d_1) except possibly $B(x, r), r \geq 1$ for any $x \in X$ is an open ball in (X, d) .
(c) Every open ball in (X, d_1) is an open ball in (X, d)
(d) None of the above.

(10) Let $(X, \| \ \|)$ be a normed linear space. Let $A \subseteq X$ and U be an open subset of X in (X, d) where d is the metric induced by $\| \ \|$. Then
(a) $A + U$ is open if and only if A is open.
(b) $A + U$ is open.
(c) $A + U$ is open if and only if $A = \emptyset$ or A is a singleton set.
(d) None of the above.

(11) Let (X, d) be a metric space , $a \in X$ and $r' > r > 0$. Let $U_1 = \{x \in X : d(x, a) > r\}, U_2 = \{x \in X : d(x, a) \neq r\}$ and $U_3 = \{x \in X : r < d(x, a) < r'\}$. Then
(a) U_1 and U_2 are open subsets of X, but U_3 may not be open.
(b) U_1, U_2, U_3 are all open.
(c) U_1 is open subset of X, but U_2 and U_3 may not be open.
(d) None of the above.

(12) Consider the metric spaces $(\mathbb{N}, d)$ and $(\mathbb{N}, d_1)$ where d is the usual distance (induced from $\mathbb{R}$) and d_1 is the discrete metric in $\mathbb{N}$. Then
(a) d and d_1 are equivalent metrics on $\mathbb{N}$, but the two metric spaces do not have same open balls.
(b) The open balls in two metric spaces are the same.
(c) Every open ball in $(\mathbb{N}, d)$ is an open ball in $(\mathbb{N}, d_1)$
(d) None of the above.

(13) Consider the following subsets of $\mathbb{C}$ with respect to the usual distance

(i) $A = \{z \in \mathbb{C} : z = 2\} \bigcup \{z \in \mathbb{C} : |z| < 2\}$

(ii) $B = \{z \in \mathbb{C} : |\text{Re } z| < a\}$ where $a > 0, a \in \mathbb{R}$

(iii) $C = \{z \in \mathbb{C} : z \neq \dfrac{i}{n}, n \in \mathbb{N}\}$

(a) A, B and C are open.
(b) B, C are open.
(c) Only B is open.
(d) Only C is open.

(14) Consider the following subsets $(\mathbb{R}^3, d)$ where d Euclidean.
$E = \{(x, y, 0) \in \mathbb{R}^3\}$
$F = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = d,$ at least one of a, b, c is not zero$\}$
$G = \{(x, y, z) \in \mathbb{R}^3 : xyz \neq 0\}$. Then
(a) E, F and G are not open.
(b) Only G is open.
(c) F, G are open.
(d) Only E is open.

(15) Let $X = C[0, 1]$ with norm $\| \ \|_\infty$.
Let $E = \{f \in X : f(0) \neq 0\}$, $F = \{f \in X : f(\frac{1}{2}) \neq 0\}$. Then
(a) E is not open and F is open.
(b) Neither E nor F are open.
(c) Both E and F are open.
(d) E is open but F is not.

(16) Let $X = C[0, 1]$. Then
(a) $B_1(0, 1)$ is open in $(X, \| \ \|_\infty)$
(b) $B_1(0, 1) \subseteq B_\infty(0, r)$ for some $r > 0$.
(c) $B_\infty(0, 1) \subseteq B_1(0, r)$ for some $r > 0$.
(d) None of the above.

Topology of Metric Spaces: Practical 3.2
Sketching of Open Balls in $\mathbb{R}^2$, Open and Closed sets, Equivalent metric spaces
Descriptive Questions 3.2

(1) Give an example of a metric space in which $B(x, r) = B(y, s)$ but $x \neq y$ and $r \neq s$.

(2) Determine which of the following sets are open in the given metric space. Justify your answer in each case.

(i) $U = \{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$ with Euclidean metric.

(ii) $U = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ with Euclidean metric.

(iii) $\mathbb{Q}$ in $\mathbb{R}$ with usual distance.

(iv) $U = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 \leq 1\}$ with Euclidean metric.

(v) $U = \{(x, y) \in \mathbb{R}^2 : 2x + 3y < 1\}$ with Euclidean metric.

(ii) $U = B((0, 0), 1) \setminus \{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3})\} \in \mathbb{R}^2$ with Euclidean metric.

(3) Let (X, d) be a discrete metric space and $x \in X$. Find
(i) $B(x, \frac{1}{2})$ (ii) $B(x, \frac{3}{4})$ (iii) $B(x, 1)$ (iv) $B(x, r), r > 1$

(4) Draw open ball $B((0, 0), 1)$ in $\mathbb{R}^2$ with respect to the given metric.

(i) d_1 induced by the norm $\| \|_1, \|x\|_1 = |x_1| + |x_2|$ for $x = (x_1, x_2) \in \mathbb{R}^2$

(ii) d_2, the Euclidean metric.

(iii) d_1 induced by the norm $\| \|_\infty, \|x\|_\infty = \max \{|x_1|, |x_2|\}$ for $x = (x_1, x_2) \in \mathbb{R}^2$

(iv) d where $d(x, y) = 2|x_1 - y_1| + 3|x_2 - y_2|$ for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$

(5) Show that in the following examples U is open subset of $(\mathbb{R}^2, d)$, where d is the Euclidean metric. Also, for $p \in U$, find maximum r_p such that $B(p, r_p) \subseteq U$.

(i) $U = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$.

(ii) $U = \{(x, y) \in \mathbb{R}^2 : x \notin \mathbb{Z}, y \notin \mathbb{Z}\}$.

(iii) $U = (0, 1) \times (0, 1)$.

(iv) $U = \{(x, y) \in \mathbb{R}^2 : -1 < x + y < 1\}$.

(6) Let $f, g \in C[0, 1]$ and suppose $f(t) < g(t)$ for each $t \in [0, 1]$. Show that $U = \{h \in C[0, 1] : f(t) < h(t) < g(t)$ for each $t \in [0, 1]\}$ is an open subset of $X = C[0, 1]$ under $\| \|_\infty$ norm where $\|f\|_\infty = \sup \{|f(t)| : t \in [0, 1]\}$

(7) Consider $X = C[0, 1]$ under the norms $\| \|_1$ and $\| \|_\infty$ where $\|f\|_1 = \int_0^1 |f(t)|\, dt$ and $\|f\|_\infty = \sup \{|f(t)| : t \in [0, 1]\}$. Draw the open ball $B(0, 1)$ in $(X, \| \|_1)$ and $(X, \| \|_\infty)$. (meaning show when does $f \in C[0, 1]$ lie in the open ball $B(0, 1)$).

(8) Describe the open balls $B(p, r)$ for $p \in \mathbb{Z}, r > 0$ considering cases $0 < r < 1, r = 1, r > 1$ in the subspace $\mathbb{Z}$ of $\mathbb{R}$ with usual distance.

(9) Let (X, d_1) and (Y, d_2) be metric spaces. Consider the metric $d : (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$ defined by $d((x_1, y_1), (x_2, y_2)) = \max \{d_1(x_1, x_2), d_2(y_1, y_2)\}$. Let $p \in X, q \in Y$ and $r, s > 0$. Show that $B(p, r) \times B(q, s)$ is an open set in $(X \times Y, d)$.

(10) Consider the metric δ on $\mathbb{R}^2$ defined by

$$\begin{aligned} \delta(x, y) &= \|x\| + \|y\| \quad \text{if } x \neq y \\ &= 0 \quad \text{if } x = y \end{aligned}$$

for $x, y \in \mathbb{R}^2$ where $\| \|$ is the Euclidean norm in $\mathbb{R}^2$. Find the open balls $B((0, 0), r)$ and $B(x, r)$ where $x \neq (0, 0), \|x\| = \epsilon$ and $0 < \epsilon < r$

(11) Check whether the following subsets of $\mathbb{C}$ with respect to usual distance are open. Justify your answer.

1. $A = \{z \in \mathbb{C} : z = 2\} \bigcup \{z \in \mathbb{C} : |z| < 2\}$
2. $B = \{z \in \mathbb{C} : |\text{Re } z| < a, \text{ where } a \in \mathbb{R}^+\}$
3. $C = \{z \in \mathbb{C} : z \neq \frac{i}{n}, n \in \mathbb{N}\}$

(12) Let (X, d) be a metric space. We define a metric d' on $X \times X$ by

$$d'((x_1, x_2), (y_1, y_2)) = \max \{d(x_1, y_1), d(x_2, y_2)\}$$

Show that $D = \{(x, x) : x \in X\}$ is a closed subset of $(X \times X, d')$

(13) Show that $S' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a closed subset of $(\mathbb{R}^2, \| \|_2), \| \|_2$ being the Euclidean metric.

(14) In the following examples, show that the given pairs of metrics are equivalent.

(i) For a metric space (X, d), the metrics d and d_1, where $d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}, x, y \in X$

(ii) For a metric space (X, d), the metrics d and d_1, where $d_1(x, y) = \min \{1, d(x, y)\}, x, y \in X$

(iii) On $\mathbb{N}, d$ and d_1 where d is the induced metric from the usual distance d in $\mathbb{R}$ and d_1 is the discrete metric.

(15) Let $X = C[0, 1]$ and d_1 and d_∞ be the metrics on X induced by $\| \|_1$ and $\| \|_\infty$. Prove or disprove d_1 and d_∞ are equivalent metrics on X.

(16) Let d_1, d_2, d_∞ be three metrics defined on $\mathbb{R}^2$ as follows:
$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$, $\quad d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
$d_\infty(x, y) = \max \{|x_1 - y_1|, |x_2 - y_2|\}, \forall x = (x_1, x_2) \quad \& \quad y = (y_1, y_2)$.
Prove that d_1, d_2, d_∞ are equivalent metrics on $\mathbb{R}^2$ by showing

$$d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{2} d_\infty(x, y) \text{ and } d_\infty(x, y) \leq d_1(x, y) \leq 2d_\infty(x, y).$$

(17) Let d_1, d_2, d_∞ be three metrics defined on $\mathbb{R}^n$ as follows:
$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$, $\quad d_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
$d_\infty(x, y) = \max \{|x_i - y_i| : 1 \leq i \leq n\}$
$\forall x = (x_1, x_2, \ldots, x_n) \quad \& \quad y = (y_1, y_2, \ldots, y_n)$
Show that $d_1(x, y) \geq d_2(x, y) \geq d_\infty(x, y) \geq n^{-\frac{1}{2}} d_2(x, y) \geq n^{-1} d_1(x, y)$

Topology of Metric Spaces: Practical 3.3
Subspaces, Interior points, Limit Points, Dense Sets and Separability, Diameter of a set, closure
Objective Questions 3.3

. (Revised Syllabus 2018-19)

(1) Consider the subspace $\mathbb{Z}$ of the metric subspace $\mathbb{R}$ with usual distance. Then
(a) Every open ball in $\mathbb{Z}$ is an infinite set.
(b) Every open ball in $\mathbb{Z}$ is a singleton set.
(c) Every open ball in $\mathbb{Z}$ is a finite set.
(d) None of the above.

(2) Let (X, d) be a metric space and $A, B \subseteq X$. Then
(a) $(A \cup B)^\circ = A^\circ \cup B^\circ, (A \cap B)^\circ = A^\circ \cap B^\circ$ (b) $(A \cup B)^\circ \subseteq A^\circ \cup B^\circ, (A \cap B)^\circ \subseteq A^\circ \cap B^\circ$
(c) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ, (A \cap B)^\circ = A^\circ \cap B^\circ$ (d) None of the above.

(3) Let A be a non-empty subset of $\mathbb{R}$, (distance being usual) then A° can be
(a) empty
(b) singleton set
(c) a finite set containing more than one element
(d) countable but not finite

(4) Consider $A = [0, 1)$ with the induced distance from the usual distance in $\mathbb{R}$. Then
(a) An open ball in A is of the type $(-r, r)$ with $0 < r < 1$
(b) $[0, \frac{1}{2})$ is an open ball in A
(c) $[0, 1)$ is not an open ball in A
(d) None of the above

(5) In the subspace $(\mathbb{Q}, d)$ of $(\mathbb{R}, d)$ where d is the usual distance in $\mathbb{R}, E = \{r \in \mathbb{Q} : 2 < r^2 < 3\}$ is
(a) an open ball (b) an open set which is not bounded.
(c) open and closed (d) None of the above.

(6) Let A be a closed subset of $\mathbb{R}$ (distance usual) $A \neq \emptyset, A \neq \mathbb{R}$. Then
(a) $A = \overline{(A^\circ)}$
(b) A is countable.
(c) A is not open.
(d) A is a bounded set.

(7) Let (X, d) be a metric space and $A, B \subseteq X$. Let $D(S)$ denote the set of limit points of $S \subseteq X$. Then
(a) If $A \subsetneq B$, then $D(A) \subsetneq D(B)$
(b) If $A \subsetneq B$, then $D(B) \subsetneq D(A)$
(c) If $A \subsetneq B$, then $D(A) \subseteq D(B)$ and the equality may occur.
(d) None of the above.

(8) Let d be the usual distance on $\mathbb{R}$ and d_1 be the discrete metric on $\mathbb{R}$. Let $A = (0,1)$. If $D(A)$ denotes the set of all limit points of A, then
(a) In $(\mathbb{R}, d), D(A) = (0,1)$ and in $(\mathbb{R}, d_1), D(A) = \{0,1\}$
(b) In $(\mathbb{R}, d), D(A) = [0,1]$ and in $(\mathbb{R}, d_1), D(A) = \emptyset$
(c) In $(\mathbb{R}, d), D(A) = (0,1)$ and in $(\mathbb{R}, d_1), D(A) = (0,1)$
(d) None of the above.

(9) Consider the following subsets of $\mathbb{R}$ (distance in $\mathbb{R}$ being usual):
(i) $\mathbb{N}$ (ii) $\mathbb{Q}$ (iii) $\{\frac{1}{n} : n \in \mathbb{N}\}$ (iv) $(-1,0)$. Then 0 is a limit point of
(a) (iv) only
(b) (ii), (iii) and (iv)
(c) (ii) and (iv) only
(d) $\mathbb{N}$

(10) Let (X,d) be a metric space and $A, B \subseteq X$. Then
(a) $\overline{A \cup B} = \overline{A} \cup \overline{B}, \overline{A \cap B} = \overline{A} \cap \overline{B}$
(b) $\overline{A \cup B} \subset \overline{A} \cup \overline{B}, \overline{A \cap B} = \overline{A} \cap \overline{B}$
(c) $\overline{A \cup B} = \overline{A} \cup \overline{B}, \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$
(d) None of the above

(11) Let (X,d) be a metric space and $A \subseteq X$. If $G \subseteq X$ is an open set such that $G \cap A = \emptyset$ then
(a) $\overline{G} \cap A = \emptyset$ (b) $G \cap \overline{A} = \emptyset$ (c) $\overline{G} \cap \overline{A} = \emptyset$ (d) None of the above

(12) Let $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \cdots\}$ in $\mathbb{R}$ where the distance is usual. Then
(a) A is a closed set. (b) A is not a closed set, $\overline{A} = (0,1]$
(c) A is not a closed set, $\overline{A} = [0,1]$
(d) None of the above.

(13) Consider $Y = [0,1] \subseteq \mathbb{R}$, with the induced usual distance d of $\mathbb{R}$. Let $A = [0,1) \subseteq Y$. Then in (Y,d)
(a) $\partial A = (0,1)$ (b) $\partial A = \{0,1\}$ (c) $\partial A = \{1\}$ (d) None of the above.

(14) Consider $\mathbb{N}$ with the induced usual distance of $\mathbb{R}$. Let $A = \{1, 2, \ldots, 10\} \subseteq \mathbb{N}$. Then the statement which is **not true** in $(\mathbb{N}, d)$ is
(a) $A^\circ = \emptyset$ (b) $\overline{A} = A$ (c) $\partial A = \emptyset$ (d) None of the above.

(15) Let $A, B \subset \mathbb{R}$, and d be the usual distance in $\mathbb{R}$. Then
(a) $d(A^\circ, B^\circ) = d(A,B) = d(\overline{A}, \overline{B})$ (b) $d(A,B) = d(\overline{A}, \overline{B})$
(c) $d(A^\circ, B^\circ) = d(A,B)$. (d) None of the above.

(16) Let (X,d) be a metric space and $A, B \subseteq X$ such that A, B are non-empty and $A \cap B = \emptyset$. Then
(a) $d(A,B) > 0$ (b) $d(A,B) > 0$ if A, B are open.
(c) $d(A,B) > 0$ if A, B are closed. (d) None of the above.

(17) Let $S^1 = \{(x,y) : x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$, distance d being Euclidean. For $p \in \mathbb{R}^2, d(p, S^1)$ equals
(a) $\|p\|$ (b) $\|p\| - 1$ (c) $\|p\| + 1$ (d) None of the above.

(18) Let $A = \{1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \cdots\}$ (distance in $\mathbb{R}$ usual). Then $\overline{A}$ equals
(a) $[0,1]$ (b) $(0,1)$ (c) $[0,1] \cap \mathbb{Q}$ (d) $\{\frac{m}{2^n}, m, n \in \mathbb{N}\} \cap [0,1]$

(19) Consider the set $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \cdots\}$ (distance in $\mathbb{R}$ usual). Then $\overline{A}$ equals
(a) A is a closed set (b) A is not a closed set, $\overline{A} = (0,1]$.
(c) A is not a closed set, $\overline{A} = [0,1]$ (d) None of the above.

(20) Let $A = \left\{\dfrac{|x|}{1+|x|} : x \in \mathbb{R}\right\}$, (distance usual). Then the set of all limit points of A is
(a) $(0,1]$ (b) $(0,\infty)$ (c) $[0,1]$ (d) None of the above.

(21) Let $A = \left\{\dfrac{x}{1+|x|} : x \in \mathbb{R}\right\}$, (distance usual). Then the set of all limit points of A is
(a) $(-1,1)$ (b) $[-1,1]$ (c) $(0,\infty)$ (d) None of the above.

Topology of Metric Spaces: Practical 3.3
Subspaces, Interior points, Limit Points, Dense Sets and Separability, Diameter of a set, closure
Descriptive Questions 3.3

(1) Give an example of a metric space $(X,d), A, B \subseteq X$ such that $A^\circ = B^\circ = \emptyset$ but $(A \cup B)^\circ = X$

(2) Find the interiors of the following subsets in a given metric space.

(i) $\mathbb{Z}$ in $(\mathbb{R}, d)$ where d is the usual distance.
(ii) $\mathbb{Q}$ in $(\mathbb{R}, d)$ where d is the usual distance.
(iii) $\{(x,y) \in \mathbb{R}^2 : x > y\} \cup \{(0,0)\}$ in $(\mathbb{R}^2, d)$ where d is the Euclidean metric.

(3) Find the closure of the following subsets of $\mathbb{C}$ (distance being usual)

(i) $S = \{z = \frac{i}{n} : n \in \mathbb{N}\}$
(ii) $S = \{z = \frac{1}{m} + \frac{i}{n} : m, n \in \mathbb{N}\}$
(iii) $S = \{z = x + iy, x, y \in (0,1), x, y \in \mathbb{Q}\}$
(iv) $S = \{z = x + iy, x, y \in (0,1)\}$

(4) Consider the subspace $A = [0,1)$ of $\mathbb{R}$ where distance in $\mathbb{R}$ is usual. Find $B_A(0,r)$ an open ball in the subspace A for $r > 0$

(5) Consider the subspace $A = [0,\infty)$ of $\mathbb{R}$ where distance in $\mathbb{R}$ is usual. Find $B_A(0,1)$ an open ball in the subspace (A,d).

(6) Show that $A = \{x \in \mathbb{Q} : -\sqrt{2} < x < \sqrt{2}\}$ is both open and closed in the subspace $\mathbb{Q}$ of $\mathbb{R}$ with usual distance.

(7) Prove or disprove : Let (X, d) be a metric space and $A \subseteq X$. Then
(i) $\overline{(A^\circ)} = \overline{A}$ (ii) $(\overline{A})^\circ = A^\circ$

(8) In $\mathbb{R}$, with respect to usual distance, show that $A = \mathbb{N}, B = \{n + \frac{1}{n} : n \in \mathbb{N}, n \neq 1\}$ are closed sets such that $A \cap B = \emptyset$. Also find $d(A, B)$.

(9) (i) In $(\mathbb{R}, d)$, where d is the usual distance, find $d(Q, \mathbb{R} \setminus \mathbb{Q})$ and $d(\mathbb{Q}, A)$ where A is any non-empty subset of $\mathbb{R}$.

(ii) In $(\mathbb{R}^2, d), d$ being Euclidean, find $d(A, B)$ where $A = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ and $B = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$.

Topology of Metric Spaces: Practical 3.4
Limit Points, Sequences, Bounded, Convergent and Cauchy Sequences in a Metric Space
Objective Questions 3.4

. (Revised Syllabus 2018-19)

(1) Let (x_n) be a sequence in a metric space $(X, d), x_n \longrightarrow p$. Let $A = \{x_n : n \in \mathbb{N}\}$. Then
(a) p is a limit point of A
(b) $p \in \overline{A}$
(c) There is a subsequence (x_{n_k}) of (x_n) having distinct terms such that $x_{n_k} \longrightarrow p$
(d) None of the above.

(2) Let S be an infinite subset of $\mathbb{R}$ such that $S \cap \mathbb{Q} = \emptyset$. Then
(a) S has a limit point which belongs to $\mathbb{R} \setminus \mathbb{Q}$.
(b) S has a limit point which belongs to $\mathbb{Q}$.
(c) S is not closed.
(d) $\mathbb{R} \setminus S$ has a limit point which is in S.

(3) Let d_1 and d_2 be equivalent metrics on X and (x_n) be a sequence in X. Then
(a) (x_n) is bounded in $(X, d_1) \iff (x_n)$ is bounded in (X, d_2).
(b) (x_n) is convergent in $(X, d_1) \iff (x_n)$ is convergent in (X, d_2).
(c) (x_n) is a Cauchy sequence in $(X, d_1) \iff (x_n)$ is a Cauchy sequence in (X, d_2).
(d) None of the above.

(4) Every Cauchy sequence is eventually constant in
(a) $(\mathbb{N}, d)$ where d is usual.
(b) $(\mathbb{Q}, d)$ where d is usual.
(c) $(\mathbb{R} \setminus \mathbb{Q}, d)$ where d is usual.
(d) None of the above.

(5) d and d_1 are metrics on $X = (0, \infty)$ where d is the usual distance and $d_1(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. Then
(a) If (x_n) is a Cauchy sequence in (X, d_1) then (x_n) is a Cauchy sequence in (X, d)
(b) If (x_n) is a Cauchy sequence in (X, d) then (x_n) is a Cauchy sequence in (X, d_1)
(c) If (x_n) is Cauchy in $(X, d_1), (x_n)$ may not be Cauchy in (X, d).
(d) (x_n) is a Cauchy sequence in $(X, d) \iff (x_n)$ is Cauchy sequence in (X, d_1)

(6) d and d_1 are metrics on $X = (0, \infty)$ where d is the usual distance and $d_1(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. Then
(a) If (x_n) is a bounded sequence in (X, d_1) then (x_n) is a bounded sequence in (X, d)
(b) If (x_n) is a bounded sequence in (X, d) then (x_n) is a bounded sequence in (X, d_1)
(c) If (x_n) is bounded in $(X, d_1), (x_n)$ may not be bounded in (X, d).
(d) (x_n) is a bounded sequence in $(X, d) \iff (x_n)$ is bounded sequence in (X, d_1)

(7) Let d_1 and d_2 be metrics on X such that $k_1 d_2(x,y) \leq d_1(x,y) \leq k_2 d_2(x,y)$ for all $x, y \in X$ where $k_1, k_2 > 0$ are constants. The the statement which is not true is
(a) (x_n) is Cauchy in (X, d_1) if and only if (x_n) is Cauchy in (X, d_2).
(b) $x_n \longrightarrow p$ in (X, d_1) if and only if $x_n \longrightarrow p$ in (X, d_2).
(c) (x_n) is bounded in (X, d_1) if and only if (x_n) is bounded in (X, d_2).
(d) None of the above.

(8) Consider the sequence (x_k) defined by $x_k = \left((-1)^k, \frac{1}{k}\right)$ in $\mathbb{R}^2$. d and d_1 are metrics on $\mathbb{R}^2$ where d is the Euclidean distance and d_1 is discrete metric. Then
(a) (x_k) is not bounded in $(\mathbb{R}^2, d)$ and $(\mathbb{R}^2, d_1)$. (b) (x_k) converges in $(\mathbb{R}^2, d)$.
(c) (x_k) has a convergent subsequence in $(\mathbb{R}^2, d)$. (d) (x_k) converges in $(\mathbb{R}^2, d_1)$.

(9) Let $x_k \longrightarrow x$ and $y_k \longrightarrow y$ in $(\mathbb{R}^n, d)$, d is Euclidean distance. Which statement is not true?
(a) $\|x_k\| \longrightarrow \|x\|$ and $\|y_k\| \longrightarrow \|y\|$.
(b) $\langle x_k, y_k \rangle \longrightarrow \langle x, y \rangle$
(c) x is a limit point of the set $A = \{x_k : k \in \mathbb{N}\}$ and y is a limit point of the set $B = \{y_k : k \in \mathbb{N}\}$
(d) $x_k + y_k \longrightarrow x + y$

(10) Consider $X = C[0,1], \|f\|_1 = \int_0^1 |f(t)| \, dt, \|f\|_\infty = \sup \{|f(t)| : t \in [0,1]\} \quad \forall f \in X$ and $f_n(x) = x^n$. Then
(a) $\{f_n\}$ converges in $(X, \| \, \|_1)$ but not in $(X, \| \, \|_\infty)$
(b) $\{f_n\}$ converges in $(X, \| \, \|_\infty)$ but not in $(X, \| \, \|_1)$
(c) $\{f_n\}$ does not converge in both.
(d) $\{f_n\}$ converges in both.

(11) Consider $(\mathbb{N}, d)$ where $d(m,n) = \begin{cases} 0 & \text{if } m = n \\ 1 + \dfrac{1}{m+n} & \text{if } m \neq n \end{cases}$ Then
(a) Every sequence in $(\mathbb{N}, d)$ is bounded.
(b) Every sequence in $(\mathbb{N}, d)$ is eventually constant.
(c) Every Cauchy sequence in $(\mathbb{N}, d)$ is eventually constant.
(d) Every sequence in $(\mathbb{N}, d)$ is Cauchy.

(12) Consider the sequence $x_n = n - [\sqrt{n}]$ in $(\mathbb{R}, d)$ where d is usual metric. Then
(a) (x_n) is Cauchy.
(b) (x_n) is monotone increasing.
(c) (x_n) is monotone decreasing.
(d) (x_n) is not convergent but has a convergent subsequence.

(13) Let d_1 and d_2 be two metrics on X and there exists real numbers $k_1, k_2 > 0$ such that $k_1 d_2(x,y) \leq d_1(x,y) \leq k_2 d_2(x,y) \quad \forall x,y \in X$. Mark the sentences which is not true.
(a) (x_n) is a Cauchy sequence in (X,d_1) implies (x_n) is a Cauchy sequence in (X,d_2)
(b) (x_n) is a bounded sequence in (X,d_1) implies (x_n) is a bounded sequence in (X,d_2)
(c) (x_n) is a convergent sequence in (X,d_1) implies (x_n) is a convergent sequence in (X,d_2)
(d) (a), (b) and (c) are not true.

(14) The sequence $\left(\frac{1}{n}\right)$ is **not** convergent in
(a) $[0,1]$ with usual distance.
(b) $[0,1]$ with discrete metric.
(c) $\mathbb{Q}$ with usual distance. (d) $[0,\infty)$ with usual distance.

(15) The Cauchy sequence which is convergent in $(\mathbb{Q},d)$, where d is the usual distance, is
(a) (x_n), where $x_n = 1 + \frac{1}{1!} + + \frac{1}{2!} \cdots + + \frac{1}{n!}$
(b) (x_n) where $x_1 = 1$ and $x_n = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$
(c) $(x_n) = \{0.1, 0.101, 0.101001, 0.1010010001, \cdots\}$ (d) (x_n) where $x_n = \frac{1}{n}\left(1 + \frac{1}{n}\right)^n$

Topology of Metric Spaces: Practical 3.4
Sequences, convergent and Cauchy sequences in a metric space
Descriptive Questions 3.4

(1) Show that the following sequences in $\mathbb{R}^2$ are convergent, distance being Euclidean.

(i) (x_n) where $x_n = \left(\frac{1}{n^2}, \frac{n^2-1}{n^3+1}\right)$

(ii) (x_n), where $x_n = \left(2^n, \frac{1}{n}\right)$ for $n \leq 9$ and $x_n = \left(2^{10}, \frac{-1}{n}\right)$ for $n \geq 10$

(2) Prove or disprove: Let d_1, d_2 be equivalent metrics on a non-empty set X. Then

(i) (x_n) is bounded in (X,d_1) if and only if (x_n) is bounded in (X,d_2)

(ii) (x_n) is Cauchy in (X,d_1) if and only if (x_n) is Cauchy in (X,d_2)

(3) Let d_1 and d_2 be equivalent metrics on a non-empty set X such that there exist $k_1, k_2 > 0$ such that

$$k_1 d_1(x,y) \leq d_2(x,y) \leq d_2 d_1(x,y) \quad \forall x,y \in X$$

Then show that

(i) (x_n) is bounded in (X,d_1) if and only if (x_n) is bounded in (X,d_2)

(ii) (x_n) is Cauchy in (X, d_1) if and only if (x_n) is Cauchy in (X, d_2)

(4) Show that the sequence $x_n = \frac{1}{n}$ converges to 0 in the usual metric space $\mathbb{R}$ but is not convergent in $X = (0, 1)$ with the usual metric.

(5) $X = C[0, 1]$. Show that $f_n(t) = e^{-nt}$ converges to 0 w.r.t. the metric $d_1(f, g) = \int_0^1 |f(x) - g(x)|\, dx$ but is not convergent w.r.t. the metric $d_\infty(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}$

(6) Let (X, d) be a metric space. If (x_n) and (y_n) are sequences in X such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$, then prove that the sequence $d(x_n, y_n) \longrightarrow d(x, y)$ in $\mathbb{R}$ w.r.t. the usual metric.

(7) Let $X = C[0, 1]$ be a metric space with the metric d_∞ defined by

$$d_\infty(f, g) = \sup\{|f(t) - g(t)| : t \in [0, 1]\}$$

Show that the sequence $\{f_n\}$ in X given by $f_n(t) = \dfrac{nt}{n + t}$ $\forall t \in [0, 1]$, is a Cauchy sequence in X.

(8) Prove that every Cauchy sequence in a discrete metric space is convergent.

(9) Let (x_n) be a Cauchy sequence in a metric space (X, d) and (x_{n_k}) be a subsequence of (x_n). Show that $d(x_n, x_{n_k}) \longrightarrow 0$ in $\mathbb{R}$ w.r.t. the usual metric.

(10) Let (x_n) and (y_n) be Cauchy sequences in a metric space (X, d). Prove that $(d(x_n, y_n))$ is a Cauchy sequence in $\mathbb{R}$ w.r.t. the usual distance.

(11) Let (X, d) be a metric space and d' be a metric on X defined by

$$d'(x, y) = \min\{1, d(x, y)\}$$

Show that (x_n) is a Cauchy sequence in (X, d) if and only if it is a Cauchy sequence in (X, d').

(12) Let (X, d_1) be a metric space and (x_n) be a sequence in X. Show that $x_n \longrightarrow x$ in (X, d_1) if and only if $d_1(x_n, x) \longrightarrow 0$ in $(\mathbb{R}, d)$ where d is the usual distance in $\mathbb{R}$.

(13) Let (a_n) and (b_n) be sequences in a metric space (X, d_1) and $x_n = d(a_n, b_n)$.If (a_n) is a Cauchy sequence in (X, d_1) and $x_n \longrightarrow 0$ in $(\mathbb{R}, d)$ (d is the usual distance), then show that (b_n) is a Cauchy sequence.

Topology of Metric Spaces: Practical 3.5
Complete Metric Spaces
Objective questions 3.5

. (Revised Syllabus 2018-19)

(1) $F_n = [n, \infty)$ for each $n \in \mathbb{N}$. Then $\cap_{n\in\mathbb{N}} F_n$
(a) has infinitely many points (b) is a singleton set.
(c) is the empty set. (d) None of the above .

(2) In $\mathbb{R}$ with respect to usual distance $\cap_{n\in\mathbb{N}} F_n$ is a singleton set when
(a) $F_n = [-n, n]$ (b) $F_n = [n, n+1]$ (c) $F_n = [1 - \frac{1}{n}, 1]$ (d) $F_n = [0, n]$

(3) $\displaystyle\bigcap_{n\in\mathbb{N}} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$ is
(a) $\{1\}$ (b) $(0, 2)$ (c) empty (d) None of these.

(4) $\displaystyle\bigcap_{n\in\mathbb{N}} (-n, n)$ is
(a) $[-1, 1]$ (b) $(-1, 1)$ (c) empty (d) None of these.

(5) $\displaystyle\bigcap_{n\in\mathbb{N}} \left[-\frac{1}{n}, \frac{1}{n}\right]$ is
(a) $\{0\}$ (b) $[-1, 1]$ (c) $[0, 1]$ (d) None of these.

(6) $\displaystyle\bigcap_{n\in\mathbb{N}} \left[0, \frac{1}{n}\right]$
(a) $\{0\}$ (b) empty (c) $[0, 1]$ (d) None of these.

(7) $f : \mathbb{R} \longrightarrow \mathbb{R}$ be any function (distance is usual). Then
(a) f is continuous on $\mathbb{R}$ if and only if f satisfies intermediate value property.
(b) If f is continuous on $\mathbb{R}$ then satisfies intermediate value property.
(c) If f satisfies intermediate value proerty and $f^{-1}(\{r\})$ is closed $\forall\ r \in \mathbb{Q}$ then f is continuous on $\mathbb{R}$.
(d) None of the above.

(8) $f : [0, 1] \longrightarrow [0, 1]$ is defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 1 - x & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1] \end{cases}$$

(a) f is continuous on $[0, 1]$ and does not satisfy intermediate value property.
(b) f satisfies intermediate value property but f is not continuous.
(c) f is continuous only at $x = \frac{1}{2}$ and $f[0, 1] = [0, 1]$. (d) None of the above.

(9) Cantor's Theorem is applicable in the following and $\cap_{n\in\mathbb{N}} F_n$ is a singleton set

(i) $X = [-1, 1], d$ usual distance, $F_n = [-\frac{1}{n}, \frac{1}{n}]$
(ii) $X = (0, 1), d$ usual distance, $F_n = [0, \frac{1}{n}]$
(iii) $X = \mathbb{R}, d$ discrete metric, $F_n = (0, \frac{1}{n})$
(iv) $X = [0, 1], d$ usual distance, $F_n = [1 - \frac{1}{n}, 1]$

(a) (i) and (ii) (b) (i) and (iv) (c) (i), (ii) and (iv) (d) None of these.

(10) Let d_1 and d_2 be equivalent metrics on X. Then
(a) (X, d_1) is bounded $\Longrightarrow (X, d_2)$ is bounded.
(b) (X, d_1) is complete $\Longrightarrow (X, d_2)$ is complete.
(c) (x_n) is a Cauchy sequence in $(X, d_1) \Longrightarrow (x_n)$ is a Cauchy sequence in (X, d_2).
(d) None of the above.

(11) Consider the following subspaces of $\mathbb{R}$ where distance in $\mathbb{R}$ is usual.
(i) $\mathbb{Q}$ (ii) $\mathbb{Z}$ (iii) $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ (iv) $[-1, 1) \cup \mathbb{N}$. Then
(a) (i) and (iv) are complete .
(b) only (ii) is complete.
(c) (ii), (iii) and (iv) are complete.
(d) None of the above.

(12) Suppose $\| \ \|_1$ and $\| \ \|_2$ are equivalent norms on a normed linear space X. Then the statement which is not true is
(a) $(X, \| \ \|_1)$ is complete if and only if $(X, \| \ \|_2)$ is complete.
(b) (x_n) is a Cauchy sequence in $(X, \| \ \|_1$ if and only if (x_n) is a Cauchy sequence in $(X, \| \ \|_2)$.
(c) A is a bounded set in $(X, \| \ \|_1)$ if and only if A is bounded in $(X, \| \ \|_2)$.
(d) (a), (b) and (c) are not true.

(13) Consider the following subspaces of $(\mathbb{R}, d)$ where d is usual distance :
(i) $[0, \infty)$ (ii) $[0, 1] \cup [2, 3]$ (iii)$\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots\}$ (iv) $\mathbb{Z}$ Then
(a) All the sub spaces are complete . (b) Only (i) is complete.
(c) Only (ii) is complete (d) Only (iii) is not complete.

(14) Let (X, d) be a complete metric space. A, B be complete subspaces of X such that $A \cap B \neq \emptyset$ then
(a) $A \cup B$ is a complete subspace of X but $A \cap B$ is not.
(b) $A \cap B$ is a complete subspace of X but $A \cup B$ is not.
(c) $A \cup B$ and $A \cap B$ are complete subspaces of X.
(d) None of the above.

(15) Consider the following subspaces under usual distance in $\mathbb{R}$.
(i) $\{\sqrt{2}, \sqrt{3}, \sqrt{5}\}$ (ii) $\{\sqrt{p} : p$ is a prime number$\}$ (iii) $\{x \in \mathbb{R} \setminus \mathbb{Q} : x \leq \sqrt{89}\}$ Then
(a) (i), (ii), (iii) are not complete.
(b) (i), (ii), (iii) are all complete.

(c) (i) and (ii) are complete and (iii) is not.
(d) None of the above.

(16) Consider the following subspaces of $(\mathbb{R}, d)$, where d is usual distance in $\mathbb{R}$. If $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}$ are subspaces of $(\mathbb{R}, d)$. Then
(a) $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are complete , $\mathbb{R} \setminus \mathbb{Q}$ is not complete.
(b) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are all complete.
(c) $\mathbb{N}, \mathbb{Z}$ are complete and $\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}$ are not complete.
(d) None of the above.

(17) Consider the space $C[a, b]$ with norms $\| \|_1$ and $\| \|_\infty$ where $\|f\|_1 = \displaystyle\int_a^b |f(x)|\, dx$ and $\|f\|_\infty = \sup\{|f(x)| x \in [a, b]\}$. Then
(a) $(C[a, b], \| \|_1)$ and $(C[a, b], \| \|_\infty)$ are complete.
(b) $(C[a, b], \| \|_1)$ is complete but $(C[a, b], \| \|_\infty)$ is not complete.
(c) $(C[a, b], \| \|_\infty)$ is complete but $(C[a, b], \| \|_1)$ is not complete.
(d) None of the above.

Topology of Metric Spaces: Practical 3.5
Complete Metric Spaces
DESCRIPTIVE QUESTIONS 3.5

(1) Check whether Cantor's Intersection theorem is applicable for the following examples. Also, find $\cap_{n\in\mathbb{N}} F_n$ in each case, where (F_n) is a sequence of subsets of $\mathbb{R}$ and the distance in $\mathbb{R}$ is usual.
(a) $Fn = (0, \infty)$ (b) $F_n = (0, \frac{1}{n})$ (c) $F_n = [1 - \frac{1}{n}, 2 + \frac{1}{n}]$

(2) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function which satisfies intermediate value property: for $a, b \in \mathbb{R}$ with $f(a) < \lambda < f(b)$, there exists c between a and b such that $f(c) = \lambda$. Further if $\{x \in \mathbb{R} : f(x) = r\}$ is closed set for each $r \in \mathbb{Q}$, then show that f is continuous on $\mathbb{R}$.

(3) Prove that there is no continuous function $f : [0, 1] \longrightarrow \mathbb{R}$ satisfying $x \in \mathbb{Q} \Longleftrightarrow f(x) \notin \mathbb{Q}$.

(4) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $f^{-1}(\{x\})$ has exactly two points for each $x \in \mathbb{R}$. Show that f cannot be continuous on $\mathbb{R}$.

(5) Let h be defined on $[0, 1]$ (usual distance) as follows:

$$h(x) = \begin{cases} 0 & \text{if} \quad x \text{ is irrational.} \\ \frac{1}{n} & \text{if} \quad x \text{ is rational number} \frac{m}{n}, \text{with} (m, n) = 1 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

Prove that h is continuous only at irrational points in $[0, 1]$.

(6) $f : [0,1] \longrightarrow [0,1]$ is defined by

$$f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \cap [0,1] \\ 1-x & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \cap [0,1] \end{cases}$$

Show that $f([0,1]) = [0,1]$ whereas f does not satisfy intermediate value property.

(7) Show that the equation $\cos x = x$ has at least one solution.

(8) Show that the equation $x^3 - 15x + 1 = 0$ has 3 solutions in the interval $[-4,4]$.

(9) Show that the function $f(x) = (x-a)^2(x-b)^2 + x$ takes the value $(a+b)/2$ for some value of x.

(10) Let $f(x) = \tan x$; then $f(\pi/4) = 1$ and $f(3\pi/4) = -1$. But there is no $c \in (\pi/4, 3\pi/4)$ such that $f(c) = 0$. Explain why this does not contradicts Intermediate value property.

(11) Prove that if f, g are continuous on $[a,b]$ and $f(a) > g(a)$ and $f(b) < g(b)$ then there is a point $c \in (a,b)$ such that $f(c) = g(c)$.

(12) Use the intermediate value property to show that there is a square whose diagonal has length between r and $2r$ and has area equal to half the area of the circle of radius r.

(13) Show that a Cauchy sequence in a metric space (X,d) where, X is a finite set and d is any distance, is eventually constant. Hence show that (X,d) is complete.

(14) Show that Cauchy sequence in $(\mathbb{N}, d)$ (or $(\mathbb{Z}, d)$) where d is usual distance is eventually constant. Hence show that $(\mathbb{N}, d)$ (or $(\mathbb{Z}, d)$) is complete.

(15) Show that a Cauchy sequence in a discrete metric space (X,d) is eventually constant. Deduce that (X,d) is complete.

(16) Show that $(\mathbb{R}^2, d)$ is a complete metric space where $d(x,y) = 2|x_1 - y_1| + 3|x_2 - y_2|$ for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

(17) Show that $(\mathbb{N}, d)$ is a complete metric space where for $m, n \in \mathbb{N}$,

$$d(m,n) = \begin{cases} 0 & \text{if } m = n \\ 1 + \dfrac{1}{m+n} & \text{if } m \neq n \end{cases}$$

(18) Let (X_1, d_1) and (X_2, d_2) be metric spaces and d be a metric on $X_1 \times X_2$ defined by $d\Big((x_1, x_2), (y_1, y_2)\Big) = \sqrt{d_1^2(x_1, y_1) + d_2^2(x_2, y_2)}$. Show that $\Big(x_n\Big) = \Big(x_1(n), x_2(n)\Big)$ in $X_1 \times X_2$ converges to (p_1, p_2) if and only if $x_1(n) \longrightarrow p_1$ and $x_2(n) \longrightarrow p_2$. Hence prove that if X_1, X_2 are complete, then $X_1 \times X_2$ is complete.

(19) Let (X_1, d_1) and (X_2, d_2) be complete metric spaces. Show that $\Big(X_1 \times X_2, d'\Big)$ and $\Big(X_1 \times X_2, d''\Big)$ are complete metric spaces where
$d'\Big((x_1, x_2), (y_1, y_2)\Big) = \alpha d_1(x_1, y_1) + \beta d_2(x_2, y_2)$
$d''\Big((x_1, x_2), (y_1, y_2)\Big) = \sqrt{\alpha d_1^2(x_1, y_1) + \alpha d_2^2(x_2, y_2)}$. where $\alpha, \beta > 0$.

(20) Show that the metric space $(C[0, 1], d_1)$ is not complete where $d_1(f, g) = \displaystyle\int_0^1 |f(x) - g(x)|\, dx$.
Hint: Consider the sequence $\{f_n\}$ in $C[0, 1]$ defined by

$$f_n(t) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ nt - \dfrac{n}{2} + 1 & \text{if } \frac{1}{2} - \frac{1}{n} < t \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

(21) Prove that $(0, 1)$ as a subspace of $(\mathbb{R}, d)$ (d being usual distance) is not complete but is complete as a subspace of $(\mathbb{R}, d_1)$ where d_1 is discrete metric.

(22) Show that $C[0, 1]$ with $\| \ \|_\infty$ defined as $\|f\|_\infty = \sup\{|f(t)| : t \in [0, 1]\}$ is complete.

Topology of Metric Spaces: Practical 3.6
Compact Metric Spaces
Objective Questions 3.6

. (Revised Syllabus 2018-19)

(1) Let (X, d) be a metric space and $K \subseteq X$. Then
(a) K is compact. (b) K is compact if K is closed.
(c) K is compact if K is bounded. (d) K is compact if K is finite.

(2) Let (X, d) be a metric space and (x_n) be a sequence in X such that $x_n \to x_0$ as $n \to \infty$. Then
(a) $\{x_n : n \in \mathbb{N}\}$ is a compact subset of X
(b) $\{x_n : n \in \mathbb{N}\} \cup \{x_0\}$ is a compact subset of X
(c) $\{x_n : n \in \mathbb{N}\} \cup \{x_0\}$ is a compact subset of X only if (x_n) is a sequence of distinct points.
(d) None of the above.

(3) Let $\{A_n\}$ be a family of compact subset of a metric space (X, d) such that $\cap_{n\in\mathbb{N}} A_n \neq \emptyset$. Then
(a) $A_1 \cup \ldots \cup A_k, k \in \mathbb{N}$ and $\cap_{n\in\mathbb{N}} A_n$ are compact subsets of X.
(b) $A_1 \cap \ldots \cup A_k, k \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}} A_n$ are compact subsets of X.
(c) $\cup_{n\in\mathbb{N}} A_n$ and $\cap_{n\in\mathbb{N}} A_n$ are compact subsets of X
(d) None of the above.

(4) Which of the following statements is false?
(a) A compact subset of a metric space is closed and bounded.
(b) A closed and bounded subset of a metric space is compact.
(c) A finite subset of a metric space is compact.
(d) A closed subset of a compact set in a metric space is compact.

(5) Which of the following are compact sunsets in the given metric space?
(a) $[0, 1]$ in $(\mathbb{R}, d_1)$ where d_1 is discrete metric.
(b) $\mathbb{N}$ in $(\mathbb{R}, d)$ where d is usual distance.
(c) $\left\{\left(\frac{1}{n}, \frac{(-1)^n}{n}\right) : n \in \mathbb{N}\right\} \cup \{(0, 0)\}$ in $(\mathbb{R}^2, d)$ where d is Euclidean distance.
(d) $[a, b] \cap \mathbb{Q}$ where a, b are irrational numbers in $(\mathbb{Q}, d)$ where d is usual distance.

(6) Consider the following subsets of $(\mathbb{R}^2, d)$, (d being Euclidean distance)

(i) $A = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$
(ii) $B = \{(x, y) \in \mathbb{R}^2 : y^2 = x\}$
(iii) $C = \{(x, y) \in \mathbb{R}^2 : 2x^2 + 3y^2 = 100\}$ Then

(a) A, B, C are compact.
(b) B, C are compact and A is not compact.
(c) Only A, B are compact.
(d) C is compact.

(7) Let (X, d) be a metric space and $x \in X$. Let $B[x, r]$ denote the closed ball $\{y \in Y : d(x, y) \leq r\}$ Then
(a) $B[x, r]$ is compact. (b) $B[x, r]$ is compact only if $r \leq 1$.
(c) $B[x, r]$ is compact if $X = \mathbb{R}$ and d is Euclidean distance. (d) None of the above.

(8) In the metric space $(\mathbb{Z}, d)$, ($\mathbb{Z}$ is the set of integers, d is usual distance), $K \subset \mathbb{Z}$
(a) if and only if K is closed. (b) if and only if K is bounded.
(c) if and only if K has a limit point. (d) if and only if $0 \in K$.

(9) Which of the following subsets of $\mathbb{R}^3$ are compact?
(a) $\{(x, y, x) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$ (b) $\{(x, y, x) \in \mathbb{R}^3 : x^2 - y^2 - z^2 = 1\}$
(c) $\{(x, y, x) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ (d) None of the above.

(10) Which of the following subsets of $\mathbb{R}^2$ is not compact? (distance being Euclidean) (a) The ellipse $\{(x, y) \in \mathbb{R}^2 : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1\}, (a, b > 0)$
(b) The rectangular hyperbola $\{(x, y) \in \mathbb{R}^2 : xy = 1\}$
(c) The set $\{(x, y) \in \mathbb{R}^2 : x^2 + 2y^2 \leq 3^2\}$ (d) The set $\{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$

(11) In the metric space $(\mathbb{R}, d)$ (d begin usual distance)
(a) $[0, 1] \cup [2, 3]$ is compact. (b) $[0, 1] \cup (2, 3)$ is compact.
(c) $[0, 1] \cup \{x \in \mathbb{N} : x \geq 3\}$ is compact. (d) $[0, 1] \cup [2, \infty)$ is compact.

(12) Consider the following subsets of $\mathbb{R}^2$ (distance being Euclidean).

(i) $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (iii) $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}$
(ii) $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

(a) A, B, C are all compact. (b) A and B are compact, C is not compact.
(c) Only B is compact. (d) Only A is compact.

(13) Consider the following subsets of $(\mathbb{R}^n, d)$ (d being Euclidean distance)

$$\begin{aligned} A = &\quad \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + x_2 + \ldots + x_n = 0\} \\ B = &\quad \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 = 1\} \\ C = &\quad \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |x_i| \leq n \text{ for } 1 \leq i \leq n\} \\ D = &\quad \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 = x_n = 0\} \end{aligned}$$

(a) A, B, C, D are compact sets. (b) Only B and C are compact sets.
(c) Only B, C and D are compact sets. (d) None of the above.

(14) Let A, B be compact subsets of $(\mathbb{R}, d)$, (d being usual). Then the following set is not compact.
(a) $A \times B$ in $(\mathbb{R}^2, d), d$ being Euclidean (b) $A \cup B$ in $\mathbb{R}$
(c) $A \cap B$ in $\mathbb{R}$ (provided $A \cap B \neq \emptyset$). (d) $A \setminus B$ in $\mathbb{R}$ (provided $A \setminus B \neq \emptyset$).

(15) Let (x_n) be a sequence in $[0, 1]$. Then, which of the following is **not true**.
(a) (x_n) has a convergent subsequence.
(b) (x_n) is bounded but may not be convergent.
(c) (x_n) is Cauchy.
(d) (x_n) may have subsequences converging to different limits.

(16) Let A be a compact subset of $\mathbb{R}$. Then
(a) $\overline{A}$ may not be compact. (b) A° may not be compact.
(c) ∂A may not be compact. (d) None of the above.

(17) Let A be a compact subset of $\mathbb{R}$. Then which of the following statements is not true
(a) A is complete. (b) A has a limit point in $\mathbb{R}$
(c) A is closed and bounded. (d) A° and ∂A are bounded.

Topology of Metric Spaces: Practical 3.6
Compact Metric Spaces
Descriptive Questions 3.6

(1) Using definition, show that $K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ is a compact subset of $(\mathbb{R}, d)$, where d is usual distance in $\mathbb{R}$. Also find a finite subcover of the open cover $\{B(\frac{1}{n}, \frac{1}{10})\}_{n\in\mathbb{N}}$ of K.

(2) Let (X, d) be a metric space and (x_n) be a sequence in X converging to x_0. Using definition, show that $K = \{x_n : n \in \mathbb{N}\} \cup \{x_0\}$ is a compact subset of (X, d)

(3) In the following examples, show that the set is not compact b considering the given open cover of the set:

(i) $C[a, b]$ in the metric space $(C[a, b], \| \|_\infty)$, $\|f\|_\infty =$ sup $\{|f(t)| : t \in [a, b]\}$. Show that the open cover $\{B(0, n)\}_{n\in\mathbb{N}}$ of $C[a, b]$ has no finite subcover. 0 being the constant zero function).

(ii) $(0, 1)$ in the metric space $(\mathbb{R}, d), d$ being the usual distance . Show that the open cover $\{(\frac{1}{n}, 1)\}_{n\in\mathbb{N}}$ of $(0, 1)$ has no finite subcover.

(iii) $\{\frac{1}{n} : n \in \mathbb{N}\}$ in the metric space $(\mathbb{R}, d), d$ being the usual distance. Show that the open cover $\{(\frac{1}{2n}, \frac{3}{2n})\}_{n\in\mathbb{N}}$ of $\{\frac{1}{n} : n \in \mathbb{N}\}$ has no finite subcover.

(iv) $[0,1]$ in the metric space $(\mathbb{R}, d_1), d_1$ being the discrete metric. Show that the open cover $\{B(x, \frac{1}{2})\}_{x\in[0,1]}$ has no finite subcover.

(4) Check if the following sets are compact in the given metric space. Justify your answer.

(i) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$ in $(\mathbb{R}^2, d), d$ being Euclidean metric.

(ii) $\{(x, y) \in \mathbb{R}^2 : xy = 1\}$ in $(\mathbb{R}^2, d), d$ being euclidean metric.

(iii) $\{n + \frac{1}{n} : n \in \mathbb{N}\}$ in $(\mathbb{R}, d), d$ being usual distance.

(5) Prove or disprove:

(i) A closed and bounded subset of a metric space is compact.

(ii) A closed ball $B[x, r]$ in a metric space is compact.

(iii) A compact set in a metric space is not open.

(iv) Interior and closure of a compact set are compact.

(6) Determine which of the following subsets of $(\mathbb{R}^2, d)$, where d is Euclidean distance is compact. Justify your answer.

(i) $\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$

(ii) $\{(x, y) \in \mathbb{R}^2 : |x| \leq 1\}$

(iii) $\{(x, y) \in \mathbb{R}^2 : x \geq 1, 0 \leq y \leq \frac{1}{x}\}$

(iv) $\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}, (a, b > 0)$

(v) $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$

(7) Let A, B be compact subsets of $\mathbb{R}$, distance being usual. Show that

(i) $A + B$ is a compact subset of $\mathbb{R}$.

(ii) $A \cup B$ is a compact subset of $\mathbb{R}$.

(iii) $A \times B$ is a compact subset of $(\mathbb{R}^2, d), d$ being Euclidean distance.

(8) Show that $A = (0, 1]$ is not a compact subset of $(\mathbb{R}, d), d$ being Euclidean distance by

(i) exhibiting a sequence in A which has no convergent sequence.

(ii) exhibiting an infinite subset of A which has no limit point in A.

(9) Show that $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + 2x_2^2 + \cdots + nx_n^2 \leq (n + 1)^2\}$ is a compact subset of $(\mathbb{R}^n, d), d$ being Euclidean.

(10) If A, B are disjoint non-empty subsets of (X, d) and A is closed, B is compact then show that $d(A, B) > 0$.

(11) Consider the set $A = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ in a metric space $(\mathbb{Q}, d)$ where d is a usual metric from $\mathbb{R}$. Is the set A:

(i) closed and bounded in $(\mathbb{Q}, d)$?

(ii) compact in $(\mathbb{Q}, d)$?

(12) Show that the closed unit ball $B[0,1]$ in l^2 is not compact, where $l^2 := \{(x_n) \text{ in } \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^2 < \infty \text{ i.e. convergent }\}$; Further, for any $x = (x_n) \in l^2$; define $||x||_2 = \sqrt{\sum_{n=1}^{\infty} |x_n|^2}$. The metric on l^2 is the metric corresponding to this norm.

Topology of Metric Spaces: Practical 3.7
Miscellaneous.

.

Revised Syllabus 2018-19

UNIT 1

(1) Define a metric space (X, d) and a normed linear space $(X, \| \ \|)$. Show that on a normed linear space $d : X \times X \longrightarrow \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ is a metric.

(2) Define an open ball $B(x, r)$ in a metric space (X, d). Show that an open ball is an open set.

(3) State and prove Hausdorff property in a metric space (X, d)

(4) Show that in a metric space (X, d)

(i) an arbitrary union of open sets is open.

(ii) a finite intersection of open sets is open.

(5) Give an example to show that an arbitrary intersection of open sets need not be open.

(6) Let (X, d) be a metric space. Show that a subset G of X is open if and only if it is a union of open balls.

(7) Prove that any nonempty open subset of $\mathbb{R}$ (distance being usual) can be written as a finite or countable union of open mutually disjoint intervals.

(8) Let (X, d) be a metric space and $A \subseteq X$. Show that

(i) A° is an open set and is the larges open set contained in A.

(ii) A is open if and only if $A = A^\circ$

(9) Let (X, d) be a metric space and $A \subseteq X$. Show that

(i) $A \subseteq B \Longrightarrow A^\circ \subseteq B^\circ$

(ii) $A \cap B)^\circ = A^\circ \cap B^\circ$

(iii) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ and the inequality may be strict.

(10) Show that two metrics d and d' on a non-empty set X are equivalent if and only if for each $x \in X$, any open ball $B_d(x, r)$ contains an open ball $B_d(x, r,')$ for some $r' > 0$ and any open ball $B'_d(x, s)$ contains an open ball $B_d(x, s')$ for some $s' > 0$.

(11) Let (X, d) be a metric space and F be a subset of X. Show that the following statements are equivalent:

(i) $X \setminus F$ is open.

(ii) F contains all its limit points.

(12) Show that in a metric space (X, d) , the following statements are equivalent for a subset G of X.

(i) G is open

(ii) G does not contain any limit point of $X \setminus G$.

(13) Let (X, d) be a metric space and $A \subseteq X$. Show that

(i) $\overline{A}$ is a closed set.

(ii) A is closed if and only if $A = \overline{A}$.

(14) Let (X, d) be a metric space and $A, B \subseteq X$. Show that

(i) $A \subseteq B \Longrightarrow \overline{A} \subset \overline{B}$

(ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(iii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ and the inequality may be strict.

(15) Let (X, d) be a metric space and $A \subseteq X$. Show that $D(D(A)) \subseteq D(A)$ where $D(S)$ denotes the set of limit points of $S \subseteq X$. Hence show that $D(A)$ is closed.

(16) Bolzano-Weierstrass Theorem: Consider a metric space $(\mathbb{R}, d)$, where d is the usual metric. Prove that every infinite bounded subset of $\mathbb{R}$ must have a limit point in $\mathbb{R}$.

UNIT II

(1) Let (X, d) be a metric space and $A \subseteq X$. Show that $p \in \overline{A}$ if and only if there is a sequence of points in A converging to p.

(2) Let (X, d) be a metric space and A be a subset of X. Show that p is a limit point of A if and only if there is a sequence of distinct points converging to p.

(3) Prove: Every bounded sequence in $\mathbb{R}$ with usual metric, has a convergent subsequence.

(4) Show that a sequence (x_k) in $(\mathbb{R}^n, d)$ (where d is Euclidean distance) converge to a point $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ if and only if $x_k^i \longrightarrow p_i$, for $1 \leq n_i$ in $\mathbb{R}$ with respect to the usual distance, where $x_k = (x_k^1, x_k^2, \cdots, x_k^n)$. Hence deduce that $(\mathbb{R}^n, d)$ is a separable metric space.

(5) Let (X, d) be a metric space and Y be a non-empty subset of X. Show that

(i) A subset G of Y is open in the subspace (Y, d) if and only of $G = V \cap Y$ where V is an open set in (X, d)

(ii) A subset F of Y is closed in the subspace (Y, d) if and only if $F = H \cap Y$ where H is closed set in (X, d).

(6) Let (X, d) be a metric space. Show that a convergent sequence in (X, d) is Cauchy. Give an example to show that the converse is not true. further show that a Cauchy sequence (x_n) in (X, d) is convergent if and only if it has a convergent subsequence.

(7) Show that the metric spaces (X, d_1) and (X, d_2) are equivalent if and only if (x_n) converges to p in (X, d_1) if and only if (x_n) converges to p in (X, d_2)

(8) Let (X, d) be a metric space . Show that a subset A of X is dense in X if and only if $G \cap A \neq \emptyset$ for each non-empty open subset G of X.

(9) Let (X, d) be a metric space. If $A \subseteq X$ is dense in X and B is a non-empty open subset of X then $\overline{A \cap B} = \overline{B}$.

(10) Prove that the metric space $(\mathbb{R}, d)$ is complete where d is the usual distance.

(11) Prove that the metric space $(\mathbb{R}^2, d)$ is complete where d is the Euclidean distance.

(12) Prove that the metric space $(\mathbb{C}, d)$ is complete with respect to the distance given by $d(z_1, z_2) = |z_1 - z_2|$

(13) Show that the metric space $(C[a, b], d)$ is complete where $d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$.

(14) Let (X, d) be a metric space and (Y, d_Y) be a subspace of (X, d). If (Y, d_Y) is complete then show that Y is closed.

(15) Let (X, d) be a complete metric space. If Y is a closed subspace of X then show that the subspace (Y, d_Y) is complete.

(16) State and prove Cantor's intersection theorem in a metric space (X, d).

(17) If in a metric space (X, d), for every decreasing sequence $\{F_n\}$ of non-empty closed sets with $d(F_n) \longrightarrow 0, \cap_{n\in\mathbb{N}} F_n$ is a singleton set then prove that (X, d) is complete.

(18) Nested Interval Theorem (As a particular case of Cantor's intersection theorem): Let $J_n = [a_n, b_n]$ be a sequence of intervals in $\mathbb{R}$ such that $J_{n+1} \subseteq J_n \forall n \in \mathbb{N}$. Then show that $\bigcap\limits_{n\in\mathbb{N}} J_n \neq \emptyset$. If further we assume that $\lim\limits_{n\longrightarrow\infty} \ell(J_n) = 0$ then show that $\bigcap\limits_{n\in\mathbb{N}} J_n$ contains precisely one point.
As a consequence of Nested Interval Theorem:

(19) Show that set $\mathbb{R}$ of real numbers is uncountable.

(20) Density of rationals: Let $x < y$ be real numbers. Show that there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

(21) Intermediate Value Theorem: Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Assume that $f(a)$ and $f(b)$ are of different signs, say, $f(a) < 0$ and $f(b) > 0$. Show that there exists $c \in (a, b)$ such that $f(c) = 0$.

UNIT III

(1) Show that a compact subset of a metric space is closed and bounded. Give an example to show that a closed and bounded subset of a metric space is not compact.

(2) Prove: A closed subset of a compact metric space is compact.

(3) Let (X, d) be a metric space and K is a compact subset of X. If F is a closed subset of X then show that $F \cap K$ is compact.

(4) Suppose (X, d) is a metric space and $\mathcal{C}$ is a non-empty collection of compact subsets of X then

(i) $\bigcap\limits_{K \in \mathcal{C}} K$ is a compact subset of X.

(ii) If $\mathcal{C}$ is finite then $\bigcup\limits_{K \in \mathcal{C}} K$ is a compact subset of X.

(5) Prove that a set A in a discrete metric space (X, d) is compact if and only if A is a finite set.

(6) Consider a metric space $(\mathbb{R}, d)$ where d is usual metric, $\emptyset \neq A \subset \mathbb{R}$. Prove that A is closed and bounded if and only if A satisfy Hein-Borel property. (A set is said to satisfy Hein-Borel property if every open conver of that set admits finite subcover).
Remark: The above result can be generalised to $(\mathbb{R}^n, d)$ as follows(without proof):
A subset A of $(\mathbb{R}^n, d)$ is closed and bounded if and only if it satisfy Hein-Borel property. Hence, $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

(7) Consider a metric space $(\mathbb{R}, d)$ where d is usual metric, $\emptyset \neq A \subset \mathbb{R}$. Prove that A is closed and bounded if and only if A is sequentially compact. (A set A is said to be sequentially compact if every sequence in A has a covergent subsequence).

(8) Consider a metric space $(\mathbb{R}, d)$ where d is usual metric, $\emptyset \neq A \subset \mathbb{R}$. Prove that A is sequentially compact if and only if A satisfy Bolzano-Weierstrass property. (A set A is said to satisfy Bolzano-Weierstrass property if every non-empty, infinite subset of A has a limit point in A).

Numerical Methods 1
Secant method, Regula-Falsi method.

Numerical Methods Objective Questions 1

(1) If $f(x)$ is a polynomial function with $f(4) = 0$, $f'(4) = 7$, $f''(4) = 10$, $f'''(4) = 30$ and all other higher derivatives of $f(x)$ at $x = 4$ are zero. Then $f(x)$ is

(a) $5x^3 - 55x^2 + 207x - 268$

(b) $x^3 - 5x^2 + 20x - 26$

(c) $5x^3 - 100x^2 + 155x - 299$

(d) None of the above.

(2) The interval in which the smallest positive root of the equation $x^3 - x - 4 = 0$ lies is

(a) $(0, 1)$ (b) $(1, 2)$ (c) $(0.5, 1.5)$ (d) $(2, 3)$

(3) The negative root of the smallest magnitude of the equation $3x^3 + 10x^2 + 10x + 7 = 0$ lies in the interval

(a) $(-1, 0)$ (b) $(-3, -2)$ (c) $(-2, -1)$ (d) $(-4, -3)$

(4) Suppose p must approximate 150 with relative error at most 10^{-3}. Then the largest interval in which p must lie is

(a) $[149.85, 150.15]$

(b) $[149.8, 150.2]$

(c) $[149.5, 150.5]$

(d) None of the above.

(5) Errors present in the statement of a problem before its solution are called

(a) truncation error (b) rounding error.

(c) inherent error (d) relative error.

(6) Errors caused by using approximate results or on replacing an infinite process by a finite one is

(a) truncation error (b) rounding error.

(c) inherent error (d) relative error.

(7) The absolute error of $\dfrac{\frac{13}{4} - \frac{6}{7}}{2e - 5.4}$ using 3 digit rounding arithmetic is

(a) 0.0788 (b) 0.154 (c) 0.55 (d) None of these

(8) Which of the following statement is true:

(a) Rate of convergence of Regular-falsi method is of second order.

(b) Rate of convergence of Secant method is of second order.

(c) Rate of convergence of Secant method is of order $\frac{\sqrt{5}+1}{2}$.

(d) None of the above.

(9) If x_{k-1} and x_k are $k-1$-th and k-th approximations to the root of $f(x)=0$ by secant method, then the next approximation x_{k+1} is

(a) $x_k - \dfrac{x_k - x_{k-1}}{f(x_k)x_k - f(x_{k-1})x_{k-1}}$

(b) $x_k - \dfrac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$ only if $f(x_k)f(x_{k-1}) < 0$

(c) $x_k - \dfrac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k)$

(d) $x_k - \dfrac{f(x_k)x_{k-1} - f(x_{k-1})x_k}{f(x_k) - f(x_{k-1})}$

(10) Consider the equation $\cos x - xe^x = 0$. Taking $x_0 = 0$, $x_1 = 1$, then the approximations x_2, x_3, x_4 by Secant method are

(a) $x_2 = 0.3147$, $x_3 = 0.4467$, $x_4 = 0.5317$.

(b) $x_2 = 0.3147$, $x_3 = 0.4467$, $x_4 = 0.5103$.

(c) $x_2 = 0.3147$, $x_3 = 0.4467$, $x_4 = 0.4523$

(d) None of the above.

(11) The secant method of finding roots of non-linear equations falls under the category of

(a) bracketing method (b) graphical method

(c) open method (d) random method.

(12) The secant method formula, for finding the square root of a real number R from the equation $x^2 - R = 0$ is

(a) $\dfrac{x_i x_{i-1} + R}{x_i + x_{i-1}}$ (b) $\dfrac{x_i x_{i-1}}{x_i + x_{i-1}}$

(c) $\dfrac{1}{2}\left\{x_i + \dfrac{R}{x_i}\right\}$ (d) $\dfrac{2x_i^2 + x_i x_{i-1} - R}{x_i + x_{i-1}}$

(13) The next iterative value of the root $x^2 - 4 = 0$ using secant method, if the initial guesses are 3 and 4 is

(a) 2.2857 (b) 2.5000

(c) 5.5000 (d) 5.7143

(14) The root of the equation $f(x) = 0$ is found by secant method.Given one of the initial estimates is $x_0 = 3, f(3) = 5$ and the angle of the secant line makes with $x - axis$ is 57^o, the next estimate of the root x_1 is

(a) -3.2470 (b) -0.24704

(c) 3.247 (d) 6.2470

(15) For finding the root of $\sin x = 0$ by the secant method, the following choices of initial guesses would not be appropriate

(a) $\dfrac{\pi}{4}$ and $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{4}$ and $\dfrac{3\pi}{4}$

(c) $\dfrac{-\pi}{2}$ and $\dfrac{\pi}{2}$ (d) $\dfrac{\pi}{3}$ and $\dfrac{\pi}{2}$

(16) Let $f(x) = x^2 - 6$. With $p_0 = 3$ and $p_1 = 2$, the value of p_3 by Regula Falsi method is

(a) 2.44444

(b) 2.45454

(c) 2.44949

(d) None of the above.

(17) A solution to $x - \cos x = 0$ in the interval $[0, \pi/2]$ that is accurate to within 10^{-4} using Regula Falsi method is

(a) 0.7390835

(b) 0.6110155

(c) 1.4330021

(d) None of the above.

(18) Order of convergence of Regula Falsi method is

(a) 1.321 (b) 1.618

(c) 2.231 (d) 2.312

(19) In Regula Falsi method, the first approximation is given by

(a) $x_1 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ (b)$x_1 = \dfrac{bf(b) - af(a)}{f(b) - f(a)}$

(c) $x_1 = \dfrac{bf(a) - af(b)}{f(a) - f(b)}$ (d) $x_1 = \dfrac{af(a) - bf(b)}{f(a) - f(b)}$

(20) For finding a real root of a equation using Regula Falsi method, the curve $y = f(x)$ is replaced by

(a) Parabola (b) Circle

(c) Straight line (d) Tangent to a curve

(21) In Regula Falsi method, if a root of $f(x) = 0$ lies between x_1 and x_2 then the approximate value of the desired root is $x_1 + h$

(a) $\dfrac{(x_1 - x_2)|y_1|}{|y_1| + |y_2|}$ (b)$\dfrac{(x_2 - x_1)|y_1|}{|y_1| + |y_2|}$

(c) $\dfrac{(x_1 - x_2)|y_2|}{|y_1| + |y_2|}$ (d) $\dfrac{(x_2 - x_1)|y_2|}{|y_1| + |y_2|}$

(22) While finding the root of an equation by method of False position the number of iterations can be reduced if we start with

(a) large interval (b) smaller interval

(c) random interval (d) None of the above

(23) If $\phi(a)$ and $\phi(b)$ are of opposite signs and the real roots of the equation $\phi(x) = 0$ is found by False position method, then first approximation x_1 of the root is

(a) $\dfrac{a\phi(b) + b\phi(a)}{\phi(b) + \phi(a)}$ (b) $\dfrac{a\phi'(b) + b\phi'(a)}{\phi(b) + \phi(a)}$

(c) $\dfrac{ab\phi(a)\phi(b)}{\phi(a) - \phi(b)}$ (d) $\dfrac{a\phi(b) - b\phi(a)}{\phi(b) - \phi(a)}$

Numerical Methods Descriptive Questions 1

(1) For the equations

   (i) $x^4 - x - 10 = 0$
   (ii) $x - e^{-x} = 0$
   (iii) $x^4 - 3x^2 + x - 10 = 0$
   (iv) $e^{-x} = \sin x$.
   (v) $x = \frac{1}{2} + \sin x$.

   Determine the initial approximations to find the smallest positive root. Find the root correct to five decimal places by

   (a) Secant Method.
   (b) Regula Falsi Method

(2) Let $f(x) = e^{-x}(2x^2 + 5x + 2) + 1$. Taking $x_0 = -1$ and $x_1 = 2.2$, find a root correct to 4 decimal places using Secant method.

(3) A real root of the equation $x^3 - 5x + 1 = 0$ lies in $(0, 1)$. Perform three iterations of the Secant method. Take each iteration correct to six places of decimals.

(4) (i) Solve $5\sin^2 x - 8\cos^5 x = 0$ for the root in the interval $(0.5, 1.5)$ by Regula Falsi method.
   (ii) Find the solution to $(x - 2)^2 - \ln x = 0$, in the interval $[1, 2]$ accurate to within 10^{-4} using Secant method.

(5) Find a root of $x\cos(\frac{x}{x-2}) = 0$ using Regula Falsi method correct to 3 decimal places.

(6) Find a root of $x^2 = \frac{e^{-2x}-1}{x}$ using Regula Falsi method correct to 3 decimal places.

(7) Find a root of $e^{x^2-1} + 10\sin(2x) - 5$ using Regula Falsi method correct to 3 decimal places.

(8) Find a root of $e^x - 3x^2 = 0$ using Regula Falsi method correct to 3 decimal places.

(9) Find a root of $\tan(x) - x - 1 = 0$ using Regula Falsi method correct to 3 decimal places.

(10) Find a root of $\sin(2x) - e^{(x-1)} = 0$ using Regula Falsi method correct to 4 decimal places.

Numerical Methods 2
Fixed-point Iteration method,Newton-Raphson method

Numerical Methods Objective Questions 2

(1) The iterative method that can be used to solve the quadratic equation $2x^2 + x - 3 = 0$ is

(a) $x_{n+1} = -3 + 2x_n^2$.
(b) $x_{n+1} = \frac{3}{2x_n+1}$.
(c) All of the above.
(d) None of the above.

(2) Let $f(x) = x^2 - a$. Then the Newton-Raphson method leads to the recurrence

(a) $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$.
(b) $x_{n+1} = x_n + \frac{a}{x_n}$.
(c) $x_{n+1} = \frac{1}{2}\left(x_n - \frac{a}{x_n}\right)$
(d) None of the above.

(3) The equation $\cos\left(\dfrac{\pi(x+1)}{8}\right) + 0.148x - 0.9062 = 0$ has roots of smallest magnitude in the intervals

(a) $(-2, -1)$ and $(1, 2)$
(b) $(-1, 0)$ and $(0, 1)$
(c) $(-\frac{7\pi}{8}, -\frac{3\pi}{4})$ and $(\frac{3\pi}{4}, \frac{7\pi}{8})$
(d) None of the above

(4) Applying Newton Raphson method to find $\sqrt{18}$, taking initial approximation $x_0 = 4$ and rounding off to four decimal places, the next iterations are

(a) $x_2 = 4.5123$ and $x_3 = 4.5121$
(b) $x_2 = 4.2426$ and $x_3 = 4.2426$
(c) $x_2 = 4.5813$ and $x_3 = 4.5813$
(d) $x_2 = 4.1419$ and $x_3 = 4.1419$

(5) Newton-Raphson method applied to the equation $f(x) = c$, where c is a constant and

$$f(x) = \begin{cases} \cos x & \text{when } |x| \le 1 \\ \cos x + (x^2-1)^2 & \text{when } |x| > 1 \end{cases}$$

gives $x_n = (-1)^n$ for each n, where initial approximation is $x_0 = 1$. Then c equals

(a) $\sin 1 - 2\cos 1$
(b) $\cos 1 + 2\sin 1$
(c) $2\sin 1 - \cos 1$
(d) $\cos 1 - 2\sin 1$

(6) Let $f(x) = 1 - x^2$ and $x_0 = 0$. Then by Newton Raphson method the value of x_1 is

(a) 0.12928 (b) 0.17294 (c) 0.12478 (d) undefined

(7) Newton-Raphson method is applied to find $\frac{1}{N}$ where N is a positive integer. For the sequence of iterates x_n to converge, the initial approximation x_0 should

(a) lie between 0 and $\frac{N}{2}$
(b) lie between 0 and $\frac{2}{N}$
(c) lie between 0 and $\frac{4}{N}$
(d) lie between $\frac{N}{2}$ and N

(8) Let $f(x) = x - 2\sin x$. Then

(a) with the initial approximation $x_0 = 1.1$, the sequence $\{x_n\}$ diverges and with initial approximation 1.5, the sequence $\{x_n\}$ converges.

(b) with the initial approximation $x_0 = 1.1$, the sequence $\{x_n\}$ converges and with initial approximation 1.5, the sequence $\{x_n\}$ diverges.

(c) with the initial approximation $x_0 = 1.1$, and 1.5, the sequence $\{x_n\}$ converges.

(d) with the initial approximation $x_0 = 1.1$, and 1.5, the sequence $\{x_n\}$ diverges.

(9) Applying Newton Raphson method with $x_0 = 0.8$ to the equation $x^3 - x^2 - x + 1 = 0$ which has exact root 1, then the rate of convergence for the first three root is

(a) of first order. (b) of second order.
(c) of third order. (d) of order $\frac{\sqrt{5}+1}{2}$.

(10) Suppose the Newton Raphson method produces a sequence that converges linearly to the root $x = \alpha$ of order $p > 1$, then the Newton Raphson iteration formula

$$x_{k+1} = x_k - \frac{pf(x_k)}{f'(x_k)}$$

will produce a sequence $\{x_k\}$ that converges

(a) quadratically to p
(b) linearly to p
(c) cubically to p
(d) None of the above.

(11) Newton-Raphson method has rate of convergence of order

(a) 1 (b) 2
(c) 3 (d) None of the above.

(12) Iteration method had rate of convergence of order

(a) 1 (b) 2
(c) 3 (d) None of the above.

(13) To find the smallest root of the equation $f(x) = x^3 - x - 1 = 0$ by iteration method, $f(x) = 0$ should be rewritten as

(a) $x = x^3 - 1$ (b) $x = (x+1)^{\frac{1}{3}}$
(c) $x = \frac{1}{\sqrt{x^2-1}}$ (d) $x = \frac{x+1}{x^2}$

(14) Newton-Raphson converges if

(a) $\left|\frac{f'(x)f''(x)}{[f(x)]^2}\right| < 1$

(b) $\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$

(c) $\left|\frac{f(x)f'(x)}{[f''(x)]^2}\right| < 1$

(d) None of the above

(15) Which one of the following is not correct

(a) Newton-Raphson method has quadratic rate of convergence.

(b) To solve $f(x) = 0$ by iteration method the given equation is written in the form $x = \phi(x)$ where $|\phi'(x)| < 1$ is an interval containing root.

(c) The method of Regular-Falsi converges faster than the secant method.

(d) None of these.

Numerical Methods Descriptive Questions 2

(1) Obtain polynomial approximation to $f(x) = e^x$ (around $x = 0$) using Taylor series expansion.Find the number of terms in the approximation so that truncation error is less than 10^{-6} for $0 \leq x \leq 1$

(2) Solve the following equations using iteration method

(1) $x^2 - 4x + 2 = 0$

(2) $x^3 + 2x + 1 = 0$

(3) Find a real root of the equation $x^3 = 1 - x^2$ in the interval $[0, 1]$ with an accuracy of 10^{-4}using iteration method.

(4) Find a real root correct to three decimal places of the equation $2x - 3 = \cos(x)$ lying in the interval $[\frac{3}{2}, \frac{\pi}{2}]$ using iteration method.

(5) Show that the iterative scheme $x_{i+1} = \frac{4x_i - ax_i^4}{3}$ is of second order.Process for calculation of the cube root of a.Use the scheme to find $4^{\frac{-1}{3}}$ to four decimal places.

(6) Use the method of iteration to find a positive root of the equation $xe^x = 1$ given that root lies in $[0, 1]$

(7) Use iterative method to find a real root of the equation $\sin x = 10(x - 1)$ correct to three decimal places.

(8) Use iterative method to find a real root of the equation $2x = \cos x + 3$ correct to three decimal places.

(9) Use iterative method to find a real root of the equation $2x - \log_{10} x + 7$ correct to four decimal places.

(10) Use iterative method to find a real root of the equation $\sin^2 x = x^2 - 1$ correct to four decimal places.

(11) For the equations

(i) $x^4 - x - 10 = 0$

(ii) $x - e^{-x} = 0$

(iii) $x^4 - 3x^2 + x - 10 = 0$

(iv) $e^{-x} = \sin x$.

(v) $x = \frac{1}{2} + \sin x$.

(vi) $x \sin x + \cos x = 0$

Determine the initial approximations to find the smallest positive root. Find the root correct to five decimal places by Newton-Raphson Method

(12) Perform four iterations of the Newton Raphson method to obtain approximate value of $(17)^{1/3}$ correct to six decimal places taking initial approximation as $x_0 = 2$.

(13) The equations $2e^{-x} = \frac{1}{x+2} + \frac{1}{x+1}$ has two roots greater than -1. Calculate the roots correct to five decimal places using Newton-Raphson Method.

(14) Find all roots of $\cos x - x^2 - x = 0$ correct to five decimal places by Newton-Raphson method.

(15) Apply the Newton-Raphson method with $x_0 = 0.8$ to the equation $x^3 - x^2 - x + 1 = 0$ and verify that the convergence is of first order. Apply Newton-Raphson method $x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$ where $m =$ the multiplicity of root $(= 2)$ and verify that the convergence is of order 2.

(16) Show that the equation $f(x) = 1 - xe^{1-x}$ has a double root at $x = 1$. Obtain the root by Newton-Raphson method taking $x_0 = 0$.

(17) Find the negative root of the equation $f(x) = \cos\left(\frac{\pi(x+1)}{8}\right) + 0.148x - 0.9062 = 0$. Correct to 4 decimals by Newton-Raphson method taking $x_0 = -0.5$.

(18) The equation $f(x) = 0$ has a simple root in the interval $(1, 2)$. The function $f(x)$ is such that $|f'(x)| \geq 4$ and $|f''(x)| \leq 3 \; \forall \; x \in (1, 2)$. Assuming Newton Raphson method to converge for all initial approximations in $(1, 2)$. Find the maximum number of iterations required to obtain root correct to 6 decimal places after rounding.

(19) Using Newton-Raphson method compute the point of intersection of the curve $y = x^3$ and straight line $y = 8x + 4$ near the point $x = 3$ correct to 2 decimal places.

Numerical Analysis 3
Iteration methods based on second degree equation - Muller Method, Chebyshev Method, Multipoint iteration Methods

Numerical Methods Objective Questions 3

(1) For the equation $x^3 - 5x + 1 = 0$ taking approximations $x_0=0$, $x_1=0.5$, $x_2=1$ Muller's method gives next approximation x_3 as

(a) 0.234516 (b) 0.191857 (c) 0.282314 (d) 0.120416

(2) In Muller's method the equation f(x)=0 is approximated by

(a) a cubic polynomial passing through three points lying on f(x)=0

(b) a quadratic polynomial passing through three points lying on f(x)=0.

(c) a linear equation passing through the given points.

(d) None of the above.

(3) Using Chebyshev's, to find the root of $f(x)= x^3 - 5x + 1 = 0$ taking initial approximation $x_0=0.5$, the next approximation x_1 is

(a) 0.723145 (b) 0.213414 (c) 0.631423 (d) 0.123160

(4) Chebyshev method and Muller's method require for each iteration

(a) three function evaluations for both.

(b) two function evaluations for both.

(c) One function evaluation for Muller's method and three function evaluations for Chebyshev's method.

(d) None of the above.

(5) The equation $f(x) = \cos x - x^2 - x = 0$ has

(a) two real roots, one in the interval $(-1,0)$ and the other in the interval $(0,1)$.

(b) two real roots, one in the interval $(-2,-1)$ and the other in the interval $(0,1)$.

(c) three real roots, one in the interval $(-2,-1)$, one in the interval $(0,1)$ and the other in the interval $(\frac{3\pi}{2}, 2\pi)$

(d) None of the above.

(6) Using multipoint method to find the root of $x^3 - 5x + 1 = 0$, taking initial approximation $x_0 = 0{:}5$, we get next approximation x_1 as

(a) 0.354281 (b) 0.204732 (c) 0.532412

(7) The rate of convergence of Muller method is

(a) 1.5 (b) 1.84 (c) 2 (d) 3

(8) Using Chebyshev method to find an approximate value of $1/7$, taking initial

approximation as $x_0 = 0.1$, the next approximation x_1 is

(a) 0.092 (b) 0.112 (c) 0.139 (d) 0.214

(9) Muller's method to find a root α of $f(x) = 0$, the initial approximations x_0, x_1, x_2 satisfy

(a) $|\alpha - x_i|<0.5$, for $i = 0,1,2$.
(b) $|\alpha - x_i|<0.1$, for $i = 0,1,2$.
(c) For all initial approximations.
(d) None of the above.

(10) Consider the equation $f(x) = x^3 - 3x - 5 = 0$. Taking approximations $x_0 = 1, x_1 = 2, x_2 = 3$, the next approximation between 2 and 3 obtained by Muller method is

(a) 2.9 (b) 2.09 (c) 2.26 (d) 2.84

(11) Muller method

(a)will converge to only real roots from a real initial approximation.
(b)will converge to both real and complex roots from a real initial approximation.
(c)will converge to only complex roots from a real initial approximation.
(d)None of the above.

(12) Muller method

(a) converges cubically for both simple root and multiple root.

(b) converges quadratically for both simple root and multiple root.

(c) converges cubically for a simple root but the convergence becomes linear at a multiple root.

(d) converges quadratically for a simple root but the convergence be-comes linear at a multiple root.

Numerical Methods Descriptive Questions 3

(1) Perform five iterations of Muller method to find an approximate root of the equation $f(x) = \cos x - xe^x = 0$. Use initial approximations $x_0 = -1, x_1 = 0, x_2 = 1$.

(2) Perform two iterations with the Muller method for the following equations:
(i) $x^3 - (1/2) = 0, x_0 = 0, x_1 = 1, x_2 = 1/2$.
(ii) $ln - x + 3 = 0, x_0 = \frac{1}{4}, x_1 = \frac{1}{2}, x_2 = 1$.

(3) Use the Chebyshev method with $f(x) = x^2 - a$ and $g(x) = 1 - a/x^2$ to obtain the iteration methods converging to $\sqrt{a}$ in the form

$$x_{k+1} = \frac{1}{2}(x_k + \frac{a}{x_k}) - \frac{1}{8x_k}\left(x_k - \frac{a}{x_k}\right)^2$$

$$x_{k+1} = \frac{1}{2}x_k\,(3 - \frac{x_k^{\,2}}{a}) + \frac{3x_k}{8}(1 - \frac{x_k^{\,2}}{a})^2$$

Perform two iterations with these methods to find the value of $\sqrt{a}$

(4) Perform two iterations with the (a) Chebyshev method, (b) Multipoint iteration method for the following equations:
(i) $x^3-5x+1=0$; $x_0=0.5$,
(ii) $\cos x - xe^x=0$; $x_0=1$.
(iii) $x^4-x-10=0$; where the root lies in the interval(1,2).

(5) Perform two iterations with the Chebyshev method to find an approximate value of 1/7. Take the initial approximation as $x_0=0.1$.

(6) Use Muller method to find two iterations of the function
(i) $f(x) = (x^2-2)\sin(x^2-2)$; $x_0 = 1.2$, $x_1 = 1.3$, $x_2 = 1.4$.
(ii) $f(x) = x^6 - 7x^4 + 15x^2 - 9$, $x_0 = 1.5$, $x_1 = 1.6$, $x_2 = 1.7$.
Round off each iteration and final answer to 4 places of decimals.

(7) The multiple root α of multiplicity two to the equation

$$f(x) = 9x^4 +30x^3 + 34x^2 + 30x + 25 = 0$$

is to be determined. Take $x_0 = -1.4$ and approximations correct to 4 places of decimals using multipoint method. Verify that the rate of convergence has order 3.

(8) Perform four iterations of finding a root of the polynomial $p(x) = x^3 + 3x^2 + 5x - 7$ starting with the points x0 = 1; x1 = 2; x3 = 3 using Muller method.

Numerical Analysis 4
Bierge Vieta method &Bairstow's method

Numerical Methods Objective Questions 4

(1) The polynomial $f(x) = 4x^5 + 10x^4 - 5x^3 + 13x^2 - 6x + 2 = 0$ has

(a) has maximum of three positive and two negative roots.
(b) has maximum of four positive and one negative root.
(c) has maximum of two positive and three negative roots.
(d) None of the above.

(2) The number of real roots of the equation $4x^4 + 2x^2 - 1 = 0$ in the interval $(-1,1)$ is
(a) 4 (b) 2 (c) 0 (d) 1

(3) The number of real roots between 0 and 3 of $p(x) = x^4 - 4x^3 + 3x^2 + 4x - 4 = 0$ using sturm sequence is

(a) 4 (b) 2 (c) 0 (d) 1

(4) The number of real and complex roots of the polynomial $f(x) = x^4 - 4x^3 + 3x^2 + 4x - 4 = 0$ is

(a) 4 and 0 (b) 2 and 2
(c) 0 and 4 (d) None of these

(5) Then multiplicity of the root x = 1 of the polynomial equation $f(x) = x^5 - 2x^4 + 4x^3 - x^2 - 7x + 5 = 0$ is

(a) 0 (b) 1 (c) 2 (d) 3

(6) The sturm sequence of the polynomial equation $f(x) = x^3 - 5x + 1 = 0$ is

(a) $x^3 - 5x + 1, 3x^2 - 5, 10x - 3, 473$
(b) $3x^2 - 5, 10x - 3, 473$
(c) $-x^3 + 5x - 1, -3x^2 + 5, -10x + 3, 473$
(d) None of the above

(7) The sturm sequence of the polynomial equation $f(x) = x^3 - 5x + 1 = 0$
(a) $\mathrm{x}^3 - 5\mathrm{x} + 1, 3\mathrm{x}^2 - 5, 10\mathrm{x} - 3, 473$,
(b) $3x^2 - 5, 10x - 3, 473$
(c) $-x^3 + 5x - 1, -3x^2 + 5, -10x + 3, 473$
(d)None of the above

(7) For the polynomial $p_3(x) = x^3 + x^2 - x + 2$, taking initial approximation $p_0 = -0.9, q_0 = 0.9$, the first iteration by Bairstow's method is

(a) $p_1 = -0.9124, q_1 = 0.9231.$

(b) $p_1 = -1.0047, q_1 = 1.0031.$
(c) $p_1 = -1.2312, q_1 = 1.2516.$
(d) $p_1 = -1.5213, q_1 = 1.6123.$

(9) Consider the polynomial $f(x) = x^5 - 3x + 1$. The first iteration p_1 for the root in $(0,1)$ by Birge Vista method taking initial approximation $p_0 = 0$ is

(a) $p_1 = 0.5000$ (b) $p_1 = 0.3333$
(b) (c) $p_1 = 0.2000$ (d) $p_1 = 0.1121$

(10) Taking initial approximation as $p_0 = 0.5, q_0 = 0.5$, the quadratic factor of $x^4 + x^3 + 2x^2 + x + 1 = 0$ by Bairstow's method is

(a) $x^2 + x + 1$ (b) $x^2 - x + 1$
(c) $x^2 + 2x + 1$ (d) None of these

(11) Let $f(x) = 1 - x^2$ and $x_0 = 0$. Then by Newton Raphson method the value of x_1 is

(a) $b_1 = a_1^2, b_2 = a_2^2 - 2a_1a_2$
(b) $b_1 = a_1^2 - 2a_0a_2, b_2 = a_2^2 - 2a_1a_3 + 2a_0a_4$
(c) $b_1 = a_1^2 - a_0a_2, b_2 = a_2^2 - 2a_1a_2 + 2a_0a_3$
(d) $b_1 = a_1^2, b_2 = a_2^2 - 4a_1a_2$

(12) The method which is used to find complex roots of a polynomial is

(a) Graffe's root square method.
(b) Bairstow method.
(c) Muller method.
(d) All of the above.

Numerical Methods Descriptive Questions 4

(1) (a) Using synthetic division, find the value of $p(2), p'(2), p''(2)$ for the polynomials:
(i) $x^4 - x^3 + x - 5 = 0$
(ii) $x^5 + x^4 - 3x^2 + 2x - 7 = 0$

(b) Find multiplicity of root of

$$f(x) = x^5 - 2x^4 + 4x^3 - x^2 - 7x + 5 = 0$$

using Sturm's sequence, obtain the exact number of the real and the complex roots of the polynomials (In case of multiple roots, count the multiplicity)

(i) $x^3 + x + 1 = 0$
(ii) $2x^3 - x^2 + 2x - 1 = 0$
(iii) $4x^4 + 4x^3 + 3x^2 + 4x - 4 = 0$

(2) Use the Birge-Vieta method to find a real root correct to three decimal places of the following equations:

(i) $x^3 - 11x^2 + 35x - 22 = 0, p_0 = 0.5.$

(ii) $x^5 - x + 1 = 0, p_0 = -1.5$.
(iii) $x^6 - x^4 - x^3 - 1 = 0,\ p_0 = 1.5$

(3) Find all real roots of the equation $x^5 = 3x - 1$ correct to two decimal places using Birge Vieta method.

(4) Find correct to four significant digits, the roots of the polynomial equation

$$x^4 - 6.789x^3 + 2.995x^2 - 0.04369x + 0.00008925 = 0:$$

If the largest root is determined first and polynomial is deflated, show that the zeros of the deflated polynomial equation differ from those of the original polynomial and find their exact values.

(5) It is given that $f(x) = 9x^4 + 12x^3 + 13x^2 + 12x + 4 = 0$ has a double root near 0.5. Perform iterations to find this root by Birge-Vieta method.

(6) Given two polynomials $p(x) = x^6 - 4.8x^4 + 3.3x^2 - 0.05$ and $Q(x) = x^6 - (4.8 - h)x^4 + (3.3 + h)x^2 - (0.05 - h)$

(i) Calculate all the roots of P .

(ii) when h << 1, the roots of Q are close to those of P . Estimate the difference between the smallest positive root of P and the corresponding root of Q.

(7) Use Bairstow's method to find the roots of $z^4 - 8z^3 + 24z^2 + 336z + 120 = 0$ with the trial factor $z^2 - (\frac{9}{2})z + \frac{3}{2} = 0$ in the first instance.

(8) Show that if $x^2 + x + 1$ is an approximate quadratic factor of the polynomial $f(x) = x^3 - x - 1$, then one iteration of Bairstow's process gives the improved approximation $x^2 + 1.333x + .6667$By continuing the process further, estimate the complex zeros of the polynomial equation$f(x) = 0$.

(9) Using Birstow's method, obtain the quadratic factor of the following equations performing two iterations.

(i) $x^4 - 3x^3 + 20x^2 + 44x + 54 = 0$ with $(p, q) = (2,2)$
(ii) $x^4 - x^3 + 6x^2 + 5x + 10 = 0$ with $(p, q) = (1.14,1.42)$
(iii) $x^3 - 3.7x^2 + 6.25x - 4.069 = 0$ with $(p, q) = (-2,5)$
(iv) $x^4 - 5x^3 + 10x^2 - 10x + 4 = 0$ with $(p, q) = (0.5, -0.5)$

(10) Use initial approximation to $p_0 = 0.5, q_0 = 0.5$ to find a quadratic factor of the form $x^2 + px + q$ of the polynomial equation $x^4 + x^3 + 2x^2 + x + 1 = 0$ using Bairstow method and hence find all its roots.

(11) Use initial approximation $p_0 = 2, q_0 = 2$ to find a quadratic factor of the form $x^2 + px + q$ of the polynomial equation $x^4 - 3x^3 + 20x^2 + 44x + 54 = 0$ using Bairstow method and hence find all its roots.

Numerical Analysis 5
Direct Methods to solve System of Equations

Gauss Elimination Method, Triangularization Method, Cholesky's Method.

Numerical Analysis Objective Questions 5

(1) Consider the system of equations $AX = b$,where

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 2 & 4 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 7 \\ 24 \end{bmatrix}$$

The solution is given by
(a) $x_1 = 2,\ x_2 = 0,\ x_3 = 5$, (b) $x_1 = 2,\ x_2 = 1,\ x_3 = 3$
(c) $x_1 = 2,\ x_2 = -1,\ x_3 = 4$, (d) None of the above.

(2) The system of equation AX=b,where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 7 \\ 24 \end{bmatrix}$$

(a) has one solution $x_1 = 2\ , x_2 = 0\ , x_3 = 1.5$,(b) is not solvable, (c) has infinitely many solutions, (d) None of the above.

(3) The goal of forward elimination steps in the Gauss elimination method is to reduce the coefficient matrix to a (an) _____________ matrix.
(a) diagonal, (b) identity, (c) lower triangular, (d) upper triangular

(4) Division by zero during forward elimination steps in Gaussian elimination of the set of equations AX=b implies the coefficient matrix A
(a) is invertible, (b) is nonsingular, (c) may be singular or nonsingular, (d) is singular

(5) Partial pivoting involve searching for
(a) the smallest coefficient of an unknown quantity amongst a system of equations.
(b) an average of smallest and largest coefficient of an unknown quantity amongst a system of equations.
(c) the largest coefficient of an unknown quantity amongst a system of equations.
(d) None of above.

(6) Pivoting equation of the following system is

(i) $-3x+2y+z= 1$

(ii) $-2y+6z= 0$

(iii) $12y+z= 37$

(a) Equation (i), (b) Equation (ii), (c) Equation (iii), (d) None of the equations.

(7) A square matrix is said to be triangular if

(a) the elements above or below the main diagonal are zero.

(b) the elements above and below the main diagonal are zero.

(c) the elements above and below the main diagonal are one.

(d) the elements above and below the main diagonal are one.

(8) A square matrix A is triangular if

(a) $a_{ij}=0$ for $i > j$ and $b_{ij}=0$ for $j > i$,

(b) $a_{ij}=0$ for $i < j$ and $b_{ij}=0$ for $j < i$,

(c) $a_{ij}=0$ for $i > j$ and $b_{ij}=0$ for $j < i$,

(d) $a_{ij}=0$ for $i < j$ and $b_{ij}=0$ for $j > i$,

(9) The LU decomposition method is computationally more efficient than Naïve Gauss elimination for solving

(a) a single set of simultaneous linear equations.

(b) multiple sets of simultaneous linear equations with different coefficient matrices and the same right hand side vectors.

(c) multiple sets of simultaneous linear equations with the same coefficient matrix and different right hand side vectors.

(d) less than ten simultaneous linear equations.

(10) The u_{22} and u_{23} of upper triangular matrix U in the LU decomposition of the matrix given below

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

is

(a) 2, 5/2, (b) 1/2, 2/5, (c) 5/2, 1/2 (d) 1/2, 5/2

(11) The lower triangular matrix L in the LU decomposition of the matrix given below

$$\begin{bmatrix} 25 & 5 & 4 \\ 10 & 8 & 16 \\ 8 & 12 & 22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

is

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0.40000 & 1 & 0 \\ 0.32000 & 1.7333 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 25 & 5 & 4 \\ 0 & 6 & 14.400 \\ 0 & 0 & -4.2400 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 10 & 1 & 0 \\ 8 & 12 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0.40000 & 1 & 0 \\ 0.32000 & 1.5000 & 1 \end{bmatrix}$

(12) The upper triangular matrix U in the LU decomposition of the matrix given below

$$\begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 12 & 22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

is

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0.40000 & 1 & 0 \\ 0.32000 & 1.7333 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 25 & 5 & 4 \\ 0 & 6 & 14.400 \\ 0 & 0 & -4.2400 \end{bmatrix}$

(c) $\begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 0 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0.2000 & 0.16000 \\ 0 & 1 & 2.4000 \\ 0 & 0 & -4.240 \end{bmatrix}$

(13) Cholesky method to solve the system of equation $AX = b$ is applicable for a symmetric matrix A

(a) if $\det(A) > 0$, (b) if A is non singular,
(c) if A is positive definite, (d) if $A \neq 0$.

(14) The value of a for which

$$A = \begin{bmatrix} 2 & 2 & a \\ 3 & 8 & 5 \\ 1 & 6 & 10 \end{bmatrix}$$

is positive definite is

(a)$a < 25$ (b)$a > -5$ (c)$a > 5$ (d)None of these.

Numerical Methods Descriptive Questions 5

(1) Solve the following system of equations by Gauss Elimination method:

(a)
$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + 3x_2 + 3x_3 = 10$$
$$3x_1 - x_2 + 2x_3 = 13$$

(b)
$$2x_1 + x_2 + 4x_3 = 12$$
$$8x_1 - 3x_2 + 2x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$

(2) Find the solution of the following system by pivot technique of Gauss Elimination method:

$$A=\begin{bmatrix} 0.0003120 & 0.006032 \\ 0.5000 & 0.8942 \end{bmatrix},\ X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ b=\begin{bmatrix} 0.003328 \\ 0.9471 \end{bmatrix}$$

(3) Solve the following system of equations by Gauss Elimination method by selecting pivot equation:

(a)
$$\begin{aligned} x_2 + 3x_3 &= 9 \\ 2x_1 + 2x_2 - x_3 &= 8 \\ -x_1 + 5x_3 &= 8 \end{aligned}$$

(b)
$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 4 \\ -x_1 + 2x_2 - 3x_3 &= 32 \\ 3x_1 - 4x_3 &= 17 \end{aligned}$$

(c)
$$\begin{aligned} 4x_1 + 10x_2 - 2x_3 &= -20 \\ -x_1 - 15x_2 + 3x_3 &= 30 \\ 25x_2 - 5x_3 &= -50 \end{aligned}$$

(d)
$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3.8 \\ 4x_1 + 3x_2 - x_3 &= -5.7 \\ 5x_1 + x_2 + 3x_3 &= 2.8 \end{aligned}$$

(e)
$$\begin{aligned} 5x_1 + 10x_2 - 2x_3 &= -0.30 \\ 2x_1 - x_2 + x_3 &= 1.91 \\ 3x_1 + 4x_2 &= 1.16 \end{aligned}$$

(4) Solve the following system of equations by method of Triangularization:

(a)
$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 13 \\ x_1 + 5x_2 + x_3 &= 14 \\ 3x_1 + x_2 + 4x_3 &= 17 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 + 5x_3 &= -3 \\ 3x_1 + 2x_2 - 3x_3 &= 6 \end{aligned}$$

(c)
$$\begin{aligned} (1/3)\,x_1 - (1/2)x_2 + (1/4)x_3 &= 1 \\ (1/2)\,x_1 + (1/4)x_2 - (1/3)x_3 &= 1 \\ (1/6)\,x_1 - (1/4)x_2 + (1/12)x_3 &= 0 \end{aligned}$$

(d)
$$\begin{aligned} 1.2x_1 - 2.3x_2 + 3.2x_3 &= 2.72 \\ 2.3x_1 + 3.2x_2 + 1.2x_3 &= 0.39 \\ 3.2x_1 - 1.2x_2 - 2.3x_3 &= 1.60 \end{aligned}$$

(e)
$$\begin{aligned} 2x_1 + 4x_2 + 3x_3 &= 9 \\ 3x_1 + x_2 - 2x_3 &= 1 \\ x_1 - x_2 + x_3 &= 6 \end{aligned}$$

(f)
$$\begin{aligned} x_1 + x_2 + 3x_3 &= 10 \\ 3x_1 + 2x_2 + 4x_3 &= 20 \\ 3x_1 + 5x_2 - x_3 &= 30 \end{aligned}$$

(5) Solve the following system of equations by LU decomposition:

$$A=\begin{bmatrix} 1 & 1 & 1 & -2 \\ 4 & 0 & 2 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 3 & 2 & -1 \end{bmatrix} \quad b=\begin{bmatrix} -10 \\ 8 \\ 7 \\ -5 \end{bmatrix}$$

also find A^{-1}.

(6) Find the inverse of the matrix using LU decomposition with $a_{11} = a_{22} = a_{33} = 1$

$$A=\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

(7) Show that the following matrices are non-singular, but cannot be written as the product LU where L is the unit lower triangular and U upper triangular.

(a) $A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ (b) $A=\begin{bmatrix} 2 & 1 & -2 \\ 4 & 2 & 3 \\ -6 & 3 & 7 \end{bmatrix}$

(8) Factorize the matrix

$$A=\begin{bmatrix} 4 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & 5 & 3 \end{bmatrix}$$

into the product LU where L is the unit lower triangular and U upper triangular.

(9) Apply Crout's method to solve the following equations:

(a) $x_1 + x_2 + 2x_3 = 7$
$3x_1 + 2x_2 + 4x_3 = 13$
$4x_1 + 4x_2 + 2x_3 = 8$

(b) $2x_1 + 3x_2 + 2x_3 = 2$
$10x_1 + 3x_2 + 4x_3 = 16$
$3x_1 + 5x_2 + x_3 = -6$

(10) Decompose the matrix

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

by Crout's method.

(12) Solve the following systems of equations using Cholesky method.

(a) $4x_1 + 2x_2 + 14x_3 = 14$
$2x_1 + 17x_2 - 5x_3 = -101$
$14x_1 - 5x_2 + 83x_3 = 155$

(b) $9x_1 + 6x_2 + 12x_3 = 174$
$6x_1 + 13x_2 + 11x_3 = 236$
$2x_1 + 11x_2 + 26x_3 = 308$

(c) $4x_1 + 6x_2 + 8x_3 = 0$
$6x_1 + 34x_2 + 52x_3 = -160$
$8x_1 + 52x_2 + 129x_3 = -452$

(d) $4x_1 + 10x_2 + 8x_3 = 44$
$10x_1 + 26x_2 + 26x_3 = 128$
$8x_1 + 26x_2 + 61x_3 = 214$

(13) Solve the following systems of equations using Cholesky method. Also find A^{-1}

(a) $A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(b) $12x + 4yz = 15\,,\ 4x + 7y + z = 12, -x + y + 6z = 6$

(14) Find inverse of the following matrices by using Cholesky method.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 4 & 6 \\ 2 & 6 & 29 \end{bmatrix}$ (c)) $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 1 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Numerical Analysis 6
Iterative Methods to Solve
System of Equations and Eigen Value Problems

Iteration Methods: Jacobi iteration method, Gauss-Siedal method.
Eigen Value Problem: Jacobi's method, Power method.

Numerical Analysis Objective Questions 6

(1) Let

$$A = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{4} \\ -\frac{1}{3} & 0 & \frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} & 0 \end{bmatrix}$$

The spectral radius of A is

(a) Greater than 1 ,(b) equal to 1, (c) less than 1, (d) equal to $\sqrt{2}$

(2) For the matrix

$$\begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix},$$

the rotation matrix that will zero out a_{13} is

(a) $\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ 0 & 0 & 1 \end{bmatrix}$

(d) None of these

(3) A square matrix $[A]_{n\times n}$ is diagonally dominant if

(a) $|a_{ii}| \geq \sum_{\substack{j=1 \\ i\neq j}}^{n} |a_{ij}|, \; i = 1,2,\ldots,n$

(b) $|a_{ii}| \geq \sum_{\substack{j=1 \\ i\neq j}}^{n} |a_{ij}|, \; i = 1,2,\ldots,n$ and $|a_{ii}| > \sum_{\substack{j=1 \\ i\neq j}}^{n} |a_{ij}|$, for any $i = 1,2,\ldots,n$

(c) $|a_{ii}| \geq \sum_{j=1}^{n} |a_{ij}|, \; i = 1,2,\ldots,n$ and $|a_{ii}| > \sum_{j=1}^{n} |a_{ij}|$, for any $i = 1,2,\ldots,n$

(d) $|a_{ii}| \geq \sum_{j=1}^{n} |a_{ij}|, \; i = 1,2,\ldots,n$

(4) The interval, which contains the eigen values of the symmetric matrix is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 6 \end{bmatrix}$$

is

(a) $[-6, 6]$ (b) $[-7, 7]$ (c) $[-10, 10]$ (d) None of these

(5) For $\begin{bmatrix} 12 & 7 & 3 \\ 1 & 5 & 1 \\ 2 & 7 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \\ -2 \end{bmatrix}$ and using $[x_1 \quad x_2 \quad x_3] = [1 \quad 2 \quad 1]$ as the initial guess, the values of $[x_1 \quad x_2 \quad x_3]$ are found at the end of each iteration as

Iteration #	x_1	x_2	x_3
1	0.41667	1.1167	0.96818
2	0.93990	1.0184	1.0008
3	0.98908	1.0020	0.99931
4	0.99899	1.0003	1.0000

At what first iteration number would you trust at least 1 significant digit in your solution?

(a) 1 (b) 2 (c) 3 (d) 4

(6) The eigen values of the matrix are diagonal elements if the matrix is

(a) Diagonal (b) lower triangular
(c) Upper triangular (d) all of these

(7) The Strurm sequence of the matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is

(a) $f_0 = 1, f_1 = 2 - \lambda, f_{r+1} = (2-\lambda)f_r - f_{r-1}$, $r = 1,2,3$
(b) $f_0 = 1, f_1 = \lambda - 2, f_{r+1} = (\lambda - 2)f_r - f_{r-1}$, $r = 1,2,3$
(c) $f_0 = 1, f_1 = \lambda - 2, f_{r+1} = (\lambda - 2)f_r + f_{r-1}$, $r = 1,2,3$
(d) None of the above

(8) Let

$$A = \begin{bmatrix} 5 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & -0.5 & 0 \end{bmatrix},$$

Then

(a) There are 2 eigen values contained in the disc $|\lambda - 5| \leq 2$.
(b) There are 3 eigen values contained in the disc $|\lambda - 5| \leq 2$.
(c) There is exactly 1 eigen value contained in the disc $|\lambda - 5| \leq 2$.
(d) None of the above

(9) The largest eigen value of the matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

lies in the interval

(a) (4, 5) (b) (3, 4) (c) (1, 2) (d) None of the above

(10) Consider the linear equation Ax =b. Let us express A= L+D+U, where L is a lower triangular matrix, D is a diagonal matrix and U is an upper triangular matrix. All diagonal elements of L and U matrices are zero. Using this definition, we can write: $Dx = -(L+U) + b$. This yields us: $x^{(k+1)} = D^{-1}[b - (L+U)x^k]$

Which of the following iterative methods does the above expression represent?

(a) Jacobi iteration (b) Gauss-Seidel (c) Power method (d) None of the above.

(11) For the system of equations

$$\begin{aligned} 4x_1 + x_2 + x_3 &= 2 \\ x_1 + 5x_2 + 2x_3 &= -6 \\ x_1 + 2x_2 + 3x_3 &= -4 \end{aligned}$$

Taking initial approximation $x^{(0)} = (0.5\,,-0.5\,,-0.5)^t$. Jacobi iteration method gives $x^{(k+1)}$ as

(a) $$\begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{3} & 0 & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{2}{3} & 0 \end{bmatrix} x^{(k)} + \begin{bmatrix} \frac{1}{2} \\ -\frac{4}{3} \\ -\frac{6}{5} \end{bmatrix}, \quad k = 0\,,1\,,2........$$

(b) $$\begin{bmatrix} 0 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{5} & 0 & -\frac{2}{5} \\ -\frac{1}{3} & -\frac{2}{3} & 0 \end{bmatrix} x^{(k)} + \begin{bmatrix} \frac{1}{2} \\ -\frac{6}{4} \\ -\frac{4}{3} \end{bmatrix}, \quad k = 0\,,1\,,2........$$

(c) $$\begin{bmatrix} 0 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{5} & 0 & -\frac{2}{5} \\ -\frac{1}{5} & -\frac{2}{5} & 0 \end{bmatrix} x^{(k)} + \begin{bmatrix} \frac{1}{2} \\ -\frac{4}{3} \\ -\frac{3}{2} \end{bmatrix}, \quad k = 0\,,1\,,2........$$

(d) None of the above

(12) Let

$$A=\begin{bmatrix} 1 & 0 & k \\ 2 & 1 & 3 \\ k & 0 & 1 \end{bmatrix}, \; b=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \; b_1, b_2, b_3 \in \mathbf{R}$$

The necessary and sufficient condition on k so that the Jacobi method converges for solving AX = b is

(a) k > 0 (b) k < 0 (c) $|k| < 1$ (d) k > 2

(13) Which of the following statements is false :

(a) If A is strictly diagonally dominant matrix , then the Jacobi iteration scheme converges for any initial starting vector
(b) If A is strictly diagonally dominant matrix , then the Gauss-Seidel iteration scheme converges for any initial starting vector
(c) Rate of convergence of Gauss-Seidel scheme is thrice as that of the Jacobi scheme
(d) None of the above

(14) The rate of convergence of the Jacobi iteration method for solving the system of equations 3x + y + z = 2, x + 4y + 2z = −5, x + 2y + 5z = 2.

(a) 0.17 (b) 1.17 (c) 2.17 (d) none of these

(15) Consider the linear equation Ax =b. Let us express A= L+D+U, where L is a lower triangular matrix, D is a diagonal matrix and U is an upper triangular matrix. All diagonal elements of L and U matrices are zero. Using this definition, we can write: Dx= − (L+ U)+ b . This yields us: $x^{(k+1)}$=H $x^{(k)}$+c, k=0,1,2…. where H= − $(D+L)^{-1}$U and c=$(D+L)^{-1}$b.

Which of the following iterative methods does the above expression represent?

(a) Jacobi iteration (b) Gauss-Seidel (c) Power method (d) None of the above.

(16) For the system of equations

$$\begin{aligned} 2x_1 - x_2 \quad &= 7 \\ -x_1 + 2x_2 - x_3 &= 1 \\ -x_2 + 2x_3 &= 1 \end{aligned}$$

Taking initial approximation $x^{(0)} = (0\,,0\,,0)^t$. Gauss Seidal iteration method gives $x^{(k+1)}$ as

(a) $x^{(k+1)} = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/10 & 1/4 \end{bmatrix} x^{(k)} + \begin{bmatrix} 7/2 \\ 9/4 \\ 13/8 \end{bmatrix}$

(b) $x^{(k+1)} = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{bmatrix} x^{(k)} + \begin{bmatrix} 7/2 \\ 9/4 \\ 11/8 \end{bmatrix}$

(c) $x^{(k+1)} = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{bmatrix} x^{(k)} + \begin{bmatrix} 7/2 \\ 9/4 \\ 13/8 \end{bmatrix}$

(d) None of above

(17) To ensure that the following system of equations,

$$\begin{aligned} 2x_1 + 7x_2 - 11x_3 &= 6 \\ x_1 + 2x_2 + x_3 &= -5 \\ 7x_1 + 5x_2 + 2x_3 &= 17 \end{aligned}$$

converges using the Gauss-Seidel method, one can rewrite the above equations as follows:

(a) $\begin{bmatrix} 2 & 7 & -11 \\ 1 & 2 & 1 \\ 7 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 17 \end{bmatrix}$

(b) $\begin{bmatrix} 7 & 5 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17 \\ -5 \\ 6 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & 5 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 17 \end{bmatrix}$

(d) The equations cannot be rewritten in a form to ensure convergence.

(18) The eigen values of the matrix

$$\begin{bmatrix} 2 & \sqrt{2} & 4 \\ \sqrt{2} & 6 & \sqrt{2} \\ 4 & \sqrt{2} & 2 \end{bmatrix}$$

using Jacobi method is

(a) $-1, 2, 3$ (b) $-2, 4, 8$ (c) $-2, 3, 6$ (d) None of the above

(19) Using Power method, the iterations reduce the original matrix to

(a) a diagonal matrix (b) an upper triangular matrix
(c) a lower triangular matrix (d) a tridiagonal matrix

(20) The largest eigen value in the magnitude of

$$\begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}$$

with initial approximation $x_0 = [1 \;\; 1]^t$, using Power method is

(a) 1 (b) 9 (c) 10 (d) None of these

Numerical Analysis Descriptive Questions 6

(1) For the following system of equations

(i)
$$\begin{aligned} 4x + y + 2z &= 4 \\ 3x + 5y + z &= 7 \\ x + y + 3z &= 3 \end{aligned}$$

(ii)
$$\begin{aligned} 10x + 4y - 2z &= 12 \\ x - 10y - z &= -10 \\ 5x + 2y - 10z &= -3 \end{aligned}$$

(a) Obtain the Jacobi iteration scheme in the matrix form.
(b) Starting with $x^{(0)} = (0, 0, 0)^t$, iterate three times.
(c) Show that the Jacobi iteration scheme converges.

(2) For the following system of equations:

$$\begin{aligned} 6x+y+z&=20 \\ x+4y-z&=6 \\ x-y+5z&=7 \end{aligned}$$

(a) Obtain the Jacobi iteration scheme in the matrix form.
(b) Starting with $x^{(0)} = (3.3333, 1.5, 1.4)^t$, iterate three times.

(3) For the following system of equations

$$\begin{aligned} 9x_1 + x_2 + x_3 &= b_1 \\ 2x_1 + 10x_2 + 3x_3 &= b_2 \\ 3x_1 + 4x_2 + 11x_3 &= b_3 \end{aligned}$$

(a) Obtain the Jacobi iteration scheme in the matrix form.
(b) Starting with $x^{(0)} = (0, 0, 0)^t, b = (10, 19, 0)^t$, iterate three times.

(c) At each iteration, obtain maximum absolute relative error.
(d) Show that the Jacobi iteration scheme converges.

(4) Consider the system

$$\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

(a) Obtain the Jacobi iteration scheme in the matrix form.
(b) Starting with $x^{(0)} = (0, 0, 0)^t$, iterate three times.
(c) At each iteration, obtain maximum absolute relative error.
(d) Show that the Jacobi iteration scheme converges.

(5) Show that for each of the following matrices A, the system Ax = b can be solved by Jacobi iteration with guaranteed convergence.

(a) $\begin{bmatrix} 5 & -1 & 3 \\ 2 & -8 & 1 \\ -2 & 0 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} -2 & 0 & 4 \\ 2 & -8 & 1 \\ 5 & -1 & 3 \end{bmatrix}$, (c) $\begin{bmatrix} 4 & 2 & -2 \\ 0 & 4 & 2 \\ 1 & 0 & 4 \end{bmatrix}$

(6) For the following system of equations:

(i) $$\begin{bmatrix} -3 & 1 & 0 \\ 2 & -3 & 1 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$$

(ii) $$\begin{bmatrix} 5 & 1 & -2 \\ 3 & 4 & -1 \\ 2 & -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 10 \end{bmatrix}$$

(a) Set up the Gauss-Seidel iteration scheme in matrix form.
(b) Show that the iteration method is convergent and hence find its rate of convergence.
(c) Starting with $x^{(0)} = (0, 0, 0)^t$, iterate three times.

(7) Apply Gauss-Seidel method to solve the system

$$3x + y + z = 3\,, \qquad 2x + y + 5z = 5\,, \qquad x + 4y + z = 2$$

Give the condition of convergence of Gauss-Seidel's iteration method and show that it is satisfied in this case.

(8) Find the solution to the following system of equations using the Gauss-Seidel method.

$$
\begin{aligned}
12x_1 + 3x_2 - 5x_3 &= 1 \\
x_1 + 5x_2 + 3x_3 &= 28 \\
3x_1 + 7x_2 + 13x_3 &= 76
\end{aligned}
$$

(a) Starting with $x^{(0)} = (1, 0, 1)^t$, iterate two times.
(b) absolute relative approximate error at the end of each iteration
(c) Is the solution converging?

(9) Given the system of equations

$$
\begin{aligned}
3x_1 + 7x_2 + 13x_3 &= 76 \\
x_1 + 5x_2 + 3x_3 &= 28 \\
12x_1 + 3x_2 - 5x_3 &= 1
\end{aligned}
$$

find the solution using the Gauss-Seidel method. Use $(1, 0, 1)^t$ as the initial guess.

(10) Using $[x_1 \;\; x_2 \;\; x_3]^t = [1 \;\; 3 \;\; 5]^t$ as the initial guess, find the values of $[x_1, x_2, x_3]$ after three iterations in the Gauss-Seidel method for

$$
\begin{bmatrix} 12 & 7 & 3 \\ 1 & 5 & 1 \\ 2 & 7 & -11 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}
$$

(11) For

$$
\begin{bmatrix} 12 & 7 & 3 \\ 1 & 5 & 1 \\ 2 & 7 & -11 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 22 \\ 7 \\ -2 \end{bmatrix}
$$

and using $[x_1 \;\; x_2 \;\; x_3]^t = [1 \;\; 2 \;\; 1]^t$ as the initial guess, find the values of $[x_1 \;\; x_2 \;\; x_3]$ after three iterations using Gauss-Seidel method.

(12) The upward velocity of a rocket is given at three different times in the following table

Velocity vs. time data

Time, t (s)	5	8	12
Velocity, v (m/s)	106.8	177.2	279.2

The velocity data is approximated by a polynomial as

$$v(t) = a_1 t^2 + a_2 t + a_3, \qquad 5 \le t \le 12$$

Find the values of a_1, a_2, and a_3 using the Gauss-Seidel method. Assume an initial guess of the solution as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

and conduct two iterations. Is the above system of equations converging?

(13) Use Jacobi iteration to find the eigen values of the following matrices :

(i) $\begin{bmatrix} -2 & \sqrt{2} & 2 \\ -2 & 1 & 2 \\ 6 & 4 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & \sqrt{2} & 4 \\ \sqrt{2} & 6 & \sqrt{2} \\ 4 & \sqrt{2} & 2 \end{bmatrix}$

(14) Find all eigen values of the matrix

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

using Jacobi Iteration till the off-diagonal. Elements in magnitude are less than 0.0005.

(15) Use Jacobi's method to find the eigen values and eigenvectors of the matrix

(a) $\begin{bmatrix} 2 & 1 & 6 \\ 1 & 3 & 2 \\ 6 & 2 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 15.010 & 0.000 & 0.008 \\ 0.000 & 15.010 & -0.0058 \\ 0.008 & -0.0058 & 15.010 \end{bmatrix}$

(16) Determine the largest eigen values and the corresponding eigenvector of the following matrices correct to three decimal places using Power method. Take the initial approximate vector as $v^0 = [1\ 1\ 1]^t$

(i) $\begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 2 \\ 0 & 1 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix}$ (iii) $\begin{bmatrix} 7 & 6 & -3 \\ -12 & -20 & 24 \\ -6 & -12 & 16 \end{bmatrix}$

(iv) $\begin{bmatrix} 16 & 30 & -42 \\ -24 & -47 & 66 \\ -12 & -24 & 34 \end{bmatrix}$ (v) $\begin{bmatrix} -12 & -72 & -59 \\ 5 & 29 & 23 \\ -2 & -12 & -9 \end{bmatrix}$

(17) Find the largest and smallest eigen values and the corresponding eigenvectors of the matrix A = $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ take the initial approximate vector as $v^{(0)} = [1 \;\; 1 \;\; 1\,]^t$.

[Hint: The smallest eigen value of A is the largest eigen value of A^{-1}]

(18) Find the smallest eigen value in magnitude of matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Using four iterations of the power method. Also obtain the corresponding eigenvector. Take the initial approximate vector as $v^{(0)} = [1 \;\; 1 \;\; 1\,]^t$.

(19) Find the eigen value correct to two decimal places, which is nearest to 5 for the matrix.

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

using inverse power method. Also obtain the corresponding eigenvector. Take the initial approximate vector as $v^{(0)} = [1 \;\; 1 \;\; 1\,]^t$. [Hint: the eigen value of A, which is nearest to 5 is the smallest eigen value in magnitude of A – 5I. Hence, it is the largest eigen value in magnitude of $(A - 5I)^{-1}$]

Numerical Analysis 7
Miscellaneous Theoretical Questions
Unit 1

(1) (a) Derive the Newton-Raphson iteration method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad k = 1, 2, \cdots$$

(b) Suppose Newton-Raphson method produce a sequence $\{x_k\}_{k=0}^{\infty}$ that converges to the root α of the function $f(x)$. Prove that

(i) If α is a simple root, then the convergence is quadratic and

$$|\varepsilon_{k+1}| \approx \frac{|f''(\alpha)|}{2|f'(\alpha)|}|\varepsilon_k|^2 \qquad \text{for sufficiently large n.}$$

(ii) If α is a multiple root of order m, then the convergence is linear and

$$|\varepsilon_{k+1}| = \frac{m-1}{m}|\varepsilon_k| \qquad \text{for sufficiently large n.}$$

(c) Suppose that Newton-Raphson method produce a sequence $\{x_k\}_{k=0}^{\infty}$ that converges to the root α of order $m > 1$ of the function $f(x)$. Prove that the iteration method

$$x_{k+1} = x_k - \frac{mf(x_k)}{f'(x_k)}$$

has quadratic rate of convergence. Hence find the double root near $x = 1.1$ of $x^3 - 4x^2 + 5x - 2 = 0$.

(d) Show that Newton-Raphson iteration formula applied to the function $f(x) = x^2 - a$ $(a > 0)$ leads to the iteration formula

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right) \qquad x_0 > 0$$

for evaluating $\sqrt{a}$. Also considering the function $f(x) = x^p - a$, show that the sequence given by

$$x_{k+1} = \frac{1}{p}\left((p-1)x_k + \frac{a}{x_k^{p-1}}\right) \qquad x_0 > 0$$

can be used to evaluate $a^{1/p}$.

(2) (a) Derive the secant iteration formula

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k) \qquad \text{for } \ k = 1, 2, \cdots$$

to find the root of the continuous differentiable function $f(x)$.

(b) Show that the rate of convergence of the secant method is $p = \frac{1}{2}(1 + \sqrt{5})$.

(c) Prove that if r is the root of $f(x) = 0$ and if the equation is rewritten in the form $x = F(x)$ in such a way that $|F'(x)| \leq L \leq 1$ in an interval I, centered at $x = r$, then the sequence $x_n = F(x_{n-1})$ with x_0 arbitrary but in the interval I has $lim\ x_n = r$

(d) Show that the iteration formula

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{\{f(x_k)\}^2 \{f''(x_k)\}}{2\{f'(x_k)\}^3}$$

is cubically convergent formula for finding the roots of the polynomial equation $f(x) = 0$

(3) (a) Derive the Regula-Falsi iteration formula

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k), \quad f(x_k)f(x_{k-1}) < 0 \qquad \text{for } \ k = 1, 2, \cdots$$

to find the root of the continuous differentiable function $f(x)$.

(b) Show that the rate of convergence of Regula-Falsi method is linear.

Unit 2

(1) (a) Derive the Muller's iteration formula

$$x_{k+1} = x_k - \frac{2a_2}{a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}, \qquad k = 2, 3, ...$$

for finding the roots of the polynomial $f(x) = a_0x^2 + a_1x + a_2$.

(b) Show that the rate of convergence of Muller method is (approximately) 1.84.

(2) (a) Derive Chebyshev iteration formula

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{1}{2}\left(\frac{f(x_k)}{f'(x_k)}\right)^2\left(\frac{f''(x_k)}{f'(x_k)}\right), \qquad k = 1, 2, 3, ...$$

for finding the roots of the polynomial $f(x) = a_0x^2 + a_1x + a_2$, where a_0, a_1, a_2 are constants.

(b) Show that the rate of convergence Chebyshev method is 3

(3) (a) Derive the Multipoint iteration formula of Type 1:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'\left(x_k - \frac{1}{2}\frac{f(x_k)}{f'(x_k)}\right)}, \qquad k = 1, 2, 3, ...$$

for finding the roots of the polynomial $f(x) = a_0x^2 + a_1x + a_2$, where a_0, a_1, a_2 are constants.

(b) Derive the Multipoint iteration formula of Type 2:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{f\left(x_k - \frac{f(x_k)}{f'(x_k)}\right)}{f'(x_k)}, \qquad k = 1, 2, 3, ...$$

for finding the roots of the polynomial $f(x) = a_0x^2 + a_1x + a_2$, where a_0, a_1, a_2 are constants.

(4) (a) In a polynomial when do we say that a change of sign has occurred ? State the Descarte's rule of sign. Determine the number of positive and negative roots of the polynomial

$$f(x) = 4x^5 + 10x^4 - 5x^3 + 13x^2 - 6x + 2$$

using Descarte's rule of sign change.

(b) If p_k is an approximation of the root of p of the polynomial equation $p_n = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$, then show that the next approximation to the root using Birge-vieta method is $p_{k+1} = p_k - \frac{b_n}{c_{n-1}}$, $k = 0, 1, \ldots$ where b_k satisfies the recurrence relation $b_k = a_k + pc_{k-1}$ with $a_0 = b_0$ and c_k satisfies the recurrence relation $c_k = b_k + pc_{k-1}$ with $c_0 = b_0$.

(c) Discuss Bairstow process for determining the roots of an algebraic equation.

(4) Derive Newton-Raphson method for system of non-linear equations in two unknowns as $f(x, y) = 0$ and $g(x, y) = 0$.

Unit 3

(1) Explain forward and backward substitution method for the system of equation $AX = B$

(2) (a) Describe Triangularization method for solving numerically a system of linear equations.

(b) Describe Cholesky method for solving numerically a system of linear equations.

(c) Discuss the operational count for Gauss elimination method.

(3) Describe Jacobi iterative method for solving numerically a system of linear equations. Give sufficient conditions for convergence of the process.

(4) Describe Gauss-Seidel iterative method for solving numerically a system of linear equations. Give sufficient conditions for convergence of the process.

(5) (a) Prove that the iteration method of the form $x^{(k+1)} = Hx^{(x)} + c, \quad k = 0, 1, 2, \ldots$ for the solution of $Ax = b$ converges to the exact solution for any initial vector if $\|H\| < 1$

(b) If A is a strictly diagonally dominant matrix, then show that the Gauss-Seidel iteration method coverges for any initial starting vector.

(c) If A is a strictly diagonally dominant matrix, then show that the Jacobi iteration method coverges for any initial starting vector.

(d) Let L and U denote lower and upper triangular matrices obtained by triangular decomposition and consider the process

$$A = A_0 = L_0U_0, A_1 = U_0L_0 = L_1U_1, \ldots, A_k = U_{k-1}L_{k-1} = L_kU_k, \ldots$$

show that A and A_k have the same eigenvalues. Also show that if $B_k = L_1L_2 \ldots L_k$ converges as $k \to \infty$, then A_k converges to an upper triangular matrix having the same eigenvalues of A.

(6) (a) Describe Jacobi iterative method for finding the eigenvalues of a symmetric matrix. Give sufficient conditions for convergence of the process.

(b) Assume that the $n \times n$ matrix has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and that they are ordered in decreasing magnitude, i.e.

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|.$$

If X_0 is chosen appropriately, then show that the sequence $\left\{X_k = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})^t\right\}$ and $\{c_k\}$ generated recursively by

$$Y_k = AX_k \qquad \text{and} \qquad X_{k+1} = \frac{1}{c_{k+1}} Y_k$$

where

$$c_{k+1} = x_j^{(k)} \qquad \text{and} \qquad x_j^{(k)} = max\left\{|x_i^k| : 1 \leq i \leq n\right\}$$

will converge to the dominant eigenvector V_1 and eigenvalue λ_1, respectively.

Practical No. 1

Congruences I

Objectives

1) The remainder obtained upon dividing the sum 1! +2! +3! + ……..(375)! By 15 is
   a) 3 b) 0 c) 1 d) None of these.
2) The remainder when the sum $1^3+2^3+3^3+ \dots\dots\dots+(99)^3+(100)^3$ divided by 4 is
   a) 3 b) 0 c) 2 d) None of these.

3) Which of the following statement is false:

(a) Cube of any integer is of the form 9k or 9k + 1 or 9k – 1

(b) Cube of any integer is of the form 7k or 7k + 1 or 7k – 1

(c) Cube of any integer is of the form 5k or 5k + 1 or 5k – 1
(d) None of the above

4) What is the remainder when 89 divides 2^{44}

a) 1 b) 13 c) 7 d) None of these.

5) Following is the Complete Residue System modulo 7.
   a) A = {-2, -9, 11, 12, 4, 18, 25}.
   b) B = { -19, -11, 4, 6, 15, 75, 84}.
   c) C = {3, 19, 19, 23, 28, 35, 42}.
   d) None of the above.

6) If S = $\{a_1, a_2, \dots\dots.a_n\}$ is a Complete Residue System mod n then

$S_1 = \{aa_1, aa_2, \dots..aa_n\}$ is also a Complete Residue System mod n if.

a) (a,n) = 1 b) (a,n) > 1 c) $a>a_i$ for all a_i 's d) None of the above.

7) If $r_1, r_2, \dots.. r_{p-1}$ is any Reduced Residue System modulo a prime P then $\prod_{i=1}^{p-1} r_i$ is congruent mdulo p to

a) -1 b) p c) 1 d) None of these.

8) The number of elements in a Reduced Residue System modulo 8 is
   a) 4 b) 7 c) 8 d) None of these.

9) Which of the following set forms a Reduced Residue System mod 12

A) A = { 3,15,21,33}

B) B= { 4,20,28,44}

C) C = { 5, 25, 35, 55}

D None of the above.

10) The least positive integer congruent to 17^{35} mod 6 is

a) 1 b) 2 c) 3 d) 5

11) The remainder when 9^{250} divided by 17 is

a) 11 b) 12 c) 13 d) None of these.

12) If $x \equiv y \pmod m$ then

a) $(x, m) = (y, m)$

b) $(x, m) = 2(y,m)$

c) $(x, m) \neq (y, m)$

d) Nothing can be said

13) If p is an odd prime then $x^2 \equiv -1$ mod p has a solution if and only if

(a) $p \equiv 1$ mod 4 (b) $p \equiv 2$ mod 4 (c) $p \equiv 3$ mod 4 (d) Nothing can be said

14) If p, q are distinct primes such that for an a, $a^p \equiv a$ mod q and $a^q \equiv a$ mod p then

(a) $a^{pq} \equiv a$ mod pq (b) $a^{pq} \equiv 1$ mod pq (c) $a^{pq} \equiv 0$ mod pq d) None of these.

15) The remainder when 17! divided by 19 is

(a) 1 (b) 2 (c) 3 (d) 0

Practical No.1

Descriptive

1) If $a \equiv b \pmod{n_1}$ and $a \equiv c \pmod{n_2}$ Prove that $b \equiv c \pmod{n}$ where $n = \gcd(n_1, n_2)$

2) Prove that 97 divides $(2^{48}-1)$

3) Verify that $0,1,2,2^2, \ldots, 2^9$ form a Complete residue system modulo 11 but $0,1^2, 2^2, \ldots 10^2$ do not

4) Show that $1^2, 2^2, \ldots m^2$ is not a Complete residue system modulo m when $m>2$

5) List all the elements in a Reduced Residue System modulo 30

6) Prove that $n^{12} - 1$ is divisible by 7 if $(n, 7) = 1$.

7) Prove that $n^{13} - n$ is divisible by 2, 3, 5, 7 and 13 for any integer n.

8) Prove that $n^{12} - a^{12}$ is divisible by 13 if n and a are prime to 13.

9) Prove that $n^{12} - a^{12}$ is divisible by 91 if n and a are prime to 91.

10) Prove that $\frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n$ is an integer for every integer n.

11) What is the last digit in the ordinary decimal representation of 2^{400}?

12) What are the last two digits in the ordinary decimal representation of 3^{400}?

13) Show that 2, 4, 6,…….2m is a complete residue system modulo m if m is odd.

14) If p is an odd prime, prove that :$1^2.3^2.5^2\ldots\ldots(p-2)^2 \equiv (-1)^{(p+1)/2} \pmod{p}$,

and $2^2.4^2.6^2\ldots\ldots(p-1)^2 \equiv (-1)^{(p+1)/2} \bmod p)$.

15) If p is a prime other than 2 or 5, Prove that p divides infinitely many of the integers 9,99,999,9999,……..

16) For a prime p, and integers a, b if $a^p \equiv b^p \bmod p$ then prove that (i) $a \equiv b \bmod p$ (ii) $a^p \equiv b^p \bmod p^2$

17) If p, q are distinct primes prove that $p^{q-1} + q^{p-1} \equiv 1 \bmod pq$

18) Find a solution of $x^2 \equiv -1 \bmod 29$.

19) Use Fermat's method to factor the following numbers: 10541 , 340663, $(2^{11}) - 1$

20) Employ the generalized Fermat's method to factor the following numbers: 4573, 6923

21) Use Kraitchik's method to factor the number 20437

Practical No.2

Congruences II

Objectives

1)The value of $\phi(728)$ is

a) 288 b) 290 c) 382 d) None of these

2) The number of positive integers ≤ 1200 and relatively prime to 1200 is

a) 420 b) 320 c) 520 d) None of these

3)The number of positive integers ≤ 4500 that have a factor greater than 1 in common with 4500 is

a)3700 b) 3300 c) 1200 d) None of these

4) $\phi(3^{100})$ equals

(a) 3^{100} (b) 3^{99} (c) 2.3^{99} (d) None of these

(5) If n is an positive integer then of $\phi(2n) = 2\phi(n)$ is

(a) always true (b) Never true (c) true If n is odd (d) true if n is even

6) Which of the following is false:

(a) $\phi(x) = 12$ has no solution (b) $\phi(x) = 13$ has no solution

(c) $\phi(x) = 14$ has no solution (d) $\phi(x) = 15$ has no solution

7)The number of distinct solutions of congruence $24x \equiv 6 \pmod{108}$ is

a) 0 b) 2 c) 6 d)12

8) If $f(x) \equiv 0 \bmod p$ has exactly n solutions and $g(x) \equiv 0 \bmod p$ has no solutions
Then $f(x)g(x) \equiv 0 \bmod p$ has
(a) exactly n solutions (b) more than n solutions (c) less than n solutions

(a) Nothing can be said

9)The solution set of the congruence $20x \equiv 4 \bmod 30$ is

(a) empty (b) singleton (c) infinite (d) None of these

10) Simultaneous solution of $17x \equiv 9 \pmod 3$ and $17x \equiv 9 \pmod 4$ are

(a) 9 mod 12 (b) 7 mod 12 (c) 5 mod 12 (d) None of these

11) A common solution to the pair of congruence $x \equiv 1 \pmod 4$; $x \equiv 2 \pmod 5$ is

a) $x \equiv 13 \pmod{20}$ b) $x \equiv 12 \pmod{20}$ c) $x \equiv 17 \pmod{20}$ d) None of these.

12) If $f(x) \equiv 0 \pmod 9$ has 3 distinct solutions and $f(x) \equiv 0 \pmod 8$ has 2 distinct solutions then $f(x) \equiv 0 \pmod{72}$ has

(a) 5 solutions (b) 6 solutions (c) l solution d)Nothing can be said

Practical No.2

Congruences II

Descriptive

(1) Find $\phi(1001)$, $\phi(5040)$
(2) Solve $\phi(x) = 24$
(3) If for an integer $n > 1$ has r distinct prime factors then prove that $\phi(n) \geq \frac{n}{2^r}$
(4) If $n > 1$ is a composite number then show that $\phi(n) \leq n - \sqrt{n}$
(5) If the integer n has r distinct odd prime factors then prove that $2^r / \phi(n)$
(6) If every prime factor that divide n also divide m then prove that $\phi(nm) = n\,\phi(m)$

7) Find all incongruent solutions of the following congruences

1) $15x \equiv 25 \pmod{85}$ 2) $30x \equiv 7 \pmod{23}$ 3) $353x \equiv 254 \pmod{400}$

8) Solve each of the following sets of simultaneous congruences:

1) $x \equiv 5 \pmod{11}$; $x \equiv 14 \pmod{29}$; $x \equiv 15 \pmod{31}$

2) $x \equiv 5 \pmod 6$; $x \equiv 4 \pmod{11}$; $x \equiv 3 \pmod{17}$

3) $x \equiv 1 \pmod 4$; $x \equiv 0 \pmod 3$; $x \equiv 5 \pmod 7$

4) $2x \equiv 1 \pmod 5$; $4x \equiv 1 \pmod 7$; $3x \equiv 9 \pmod 6$; $5x \equiv 9 \pmod{11}$

9) Find the smallest positive integer having the remainder 3,11,15 when divided by 10,13,17 respectively.

10) A band of 17 pirates stole a sack of gold coins. When they tried to divide equally among them 3 coins remain. In the ensuing fight one of the pirates died. Again it was divided equally to find 10 coins were left . Again a fight resulted in killing one more pirate. This time they were able to divide the gold equally among themselves . What was the least number of gold coins they could have stolen?

11) Solve :

a) $5x^2 - 6x + 2 \equiv 0 \pmod{13}$ b) $x^2 + 7x + 10 \equiv 0 \pmod{11}$ c) $3x^2 + 9x + 7 \equiv 0 \pmod{13}$

d) $5x^2 + 6x + 1 \equiv 0 \pmod{23}$

12) Solve the Congruences:

a) $x^3 + 2x - 3 \equiv 0 \pmod{45}$ b) $x^3 - 9x^2 + 23x - 15 \equiv 0 \pmod{143}$ c) $x^3 + 4x + 8 \equiv 0 \pmod{15}$

<u>PRACTICAL NO. 3</u> (Diophantine Equations)

<u>OBJECTIVE QUESTIONS:</u>

1. Let (a,b)=g; (a,c)=d; (b,c)=e, then the equation ax+by=c has a solution if

   (i) g|c (ii) d|b (iii) e|a (iv) None of the above

2. The equation ax+by=c has a solution if and only if
   (i) (a ,b)=(a ,b, c) (ii) (a, c) = (a, b, c)
   (iii) (b, c) = (a, b, c) (iv) None of the above

3. If ax + by = c has two solutions (x_0, y_0) and (x_1, y_1) with x_1=1+ x_0 and y_1= 1+ y_0 and (a, b)=1 then
   (i) b=0 or b=1 (ii) b=-1 or b=0
   (iii) b=-1 or b=1 (iv) None of the above

4. Let A be the statement ' The equation ax + by=c is solvable' and B be the statement ' The equation ax + by = a + c is solvable', then
   (i) A is true whenever B is true but converse may not be true
   (ii) B is true whenever A is true but converse may not be true
   (iii)A is true if and only if B is true
   (iv) None of the above

5. If (a, b)=1 and a and b are of opposite signs then the equation ax + by = c has
   (i) Infinitely many solutions for c>0
   (ii) Infinitely many solutions for any value of c
   (iii) Has no integral solution whatever the value of c
   (iv) None of the above

6. Which of the statements is false?
   (i) The equation ax + by =c is solvable in integers iff (a, b, c) =(a, b)
   (ii) The equation ax + by =c is solvable in positive integers where a, b, c are positive then a + b ≤ c
   (iii) The equation ax + by =c has a solution in integers then the equation ax + by =a + c has a solution in integers.
   (iv) 101 x + 37y =3819 has two solutions in set of positive integers.

7. The equation ax + by = c has a solution in positive integers where a, b, c are positive then
   (i) a + b ≤ c (ii) a + b ≥ c
   (iii) Cannot have any solutions whatever the values of a, b, c (iv) None of the above

8. Let a, b, c be positive integers and (a, b) = 1 with $\frac{c}{ab}$ is not an integer but $\frac{c}{a}$ is an integer, then the number of solutions of ax + by = c in the set of positive integers is

   (i) $[\frac{c}{ab}]$ (ii) $[\frac{c}{b}]$ (iii) $[\frac{c}{a}]$ (iv) None of the above

9. The equation 3x + 6y =100 has

(i) 0 integral solutions (ii) 1 integral solution

(iii) 2 integral solutions (iv) Infinitely many integral solutions

10. $a_1, a_2, a_3, \dots \dots \dots \dots a_n$ are non- zero positive integers and the equation $a_1x_1 + a_2x_2 + a_3x_3 + \cdots \dots \dots a_nx_n = c$ has an integral solution. Also d=$(a_1, a_2, a_3, \dots \dots \dots \dots a_n)$. Then

(i) c|d (ii) d|c (iii) d=c (iv) None of the above

11. Let (a, b) =g, (a, c) =d, (b, c) =e then the equation ax + by =c has

(i) g number of integral solutions

(ii) d number of integral solutions

(iii) e number of integral solutions

(iv) None of the above

<u>DESCRIPTIVE QUESTIONS</u>:

1. A farmer sold chickens at Rs. 5/ each and geese at Rs 8/ each. He collected a total of Rs 99/-. Assuming that he sold at least one bird of each kind, how many of each kind did he sell?

2. Show that if a and b are co-prime positive integers, then every integer c ≥ ab has the form ax+by where x and y are non-negative integers. Also show that the integer ab-a-b does not have this form.

3.a) Determine all the solutions in positive integers of the following Diophantine equations.

i) 5 x + 3 y =52

ii) 15 x + 7 y =111

iii) 12 x + 510 y = 274

b) Determine all the integral solutions of the following Diophantine equations.

i) 903 x + 731 y = 2107

ii) 101 x + 99 y =437

4. A certain number of sixes and nines is added to give a sum of 126. If the number of sixes and nines is interchanged, the new sum is 114. How many of each were there originally?

5. When Mr. X cashed a cheque at his bank, the teller mistook the number of paise for the number of rupees and vice versa. Unaware of this , X spent 68 paise and then noticed to his surprise that he had twice the amount of the original cheque. Determine the smallest value for which the cheque could have been written.

6. There were 63 equal piles of fruits put together and 7 single fruits. They were divided evenly

among 23 travellers. What is the number of fruits in each pile?

7. Find the number of men, women and children in a company of 20 persons if together they pay 20 coins, each man paying 3, each woman 2 and each child ½ .

8. We have an unknown number of coins. If you make 77 strings then you are 50 coins short; but if you make 78 strings then it is exact. How many coins are there?

9. A father's age is 1 less than twice that of his son, and the digits AB making up the father's age are reversed in the son's age(ie. BA). Find their ages.

10. Clara wants to buy pizza and cola for her family She has 400 Rs. If we know that each pizza costs 57/- and each bottle of cola 22/-. How many of each can she buy?
Assume that there is discount for some stuff in the restaurant and pizza price is changed from 57 to 55. Then how many of each can she buy?

………..

PRACTICAL NO. 4

OBJECTIVE QUESTIONS:

1. The equation $x^2 - y^2$= n has solutions

   (a) for every integer n
   (b) only for integers of the form 4k+1
   (c) only for 2 and all integers of the form 4k+1
   (d) for all integers which are not of the form 4k+2

2. The equation $x^2 + y^2 + 1 = z^2$ has
   (a) no solution in integers
   (b) finitely many solutions in integers
   (c) infinitely many solutions
   (d) only two solutions

3. The equation $15x^2 - 7y^2$=9 has
   (a) two solutions in integers
   (b) no solution in integers
   (c) four solutions in integers
   (d) infinitely many solutions in integers.

4. For a given integer n the number of Pythagorean triples having the same first member is
   (a) less than n
   (b) equal to $\phi(n)$
   (c) greater than n
   (d) none of the above

5. The number of Pythagorean triples (x, y, z) for which
   (a) x, y, z are consecutive integers are infinitely many
   (b) x, y are consecutive integers are infinitely many
   (c) x, y, z are odd are infinitely many
   (d) none of the above

6. The equation $x^4 - y^4 = 2z^2$ has
(a) no solution in positive integers
(b) infinitely many solutions in positive integers
(c) finitely many solutions in positive integers
(d) none of the above

7. If p = $q_1^2 + q_2^2 + q_3^2$ where p, q_1, q_2, q_3 are primes then
(a) each q_i is of the form 4k+1
(b) each q_i, is of the form 4k+3
(c) at least one $q_i = 3$
(d) no q_i is even

8. Let n be a positive integer.
(a) If n is sum of three squares then 2n is also a sum of three squares
(b) If n is sum of three squares then 2n cannot be a sum of three squares
(c) If n is sum of three squares then 2n is a perfect square.
(d) n or 2n is sum of three squares.

9. Which of the following statements is false?
(a) 2^n is sum of two squares for all natural numbers n.
(b) Any prime of the form 4k+1 can be expressed as a sum of two squares.
(c) Any odd prime can be expressed as sum of two squares.
(d) If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots . p_k^{\alpha_k} q_1^{\beta_1} q_2^{\beta_2} \ldots \ldots . q_r^{\beta_r}$ where p_i's are primes of the form 4t+1 and q_j's are primes of the form 4t+3 then β_j's are even.

10. An integer of the form 8m + 7
(a) can be expressed as sum of two squares
(b) can be expressed as sum of three squares.
(c) cannot be expressed as sum of three squares.
(d) may be a square.

11. If n is a sum of two squares then
(a) 2n is also a sum of two squares
(b) 2n is not a sum of two squares.
(c) 2n is sum of two squares if and only if n is odd.
(d) none of the above.

DESCRIPTIVE QUESTIONS:

1)Show that the area of a Pythagorean triangle can never be equal to a perfect square.

2) If x, y, z is a primitive Pythagorean triple , prove that x + y and x - y are congruent modulo 8 to either 1 or 7.

3) Prove that in a primitive Pythagorean triple x, y, z the product xy is divisible by 12 , hence 60|xyz.

4) Show that 3n, 4n, 5n where n=1, 2,……… are the only Pythagorean triples whose terms are in Arithmetic Progression.

5)Show that the radius of the inscribed circle of a Pythagorean triangle is always an integer.

6) Show that a positive integer can be represented as the difference of two squares if and only if n is not of the form 4k+2.

7) Establish each of the following:

i) Each of the integers 2^n, where n= 1,2,…… is a sum of two squares.

ii) If n≡ 3 or 6 (mod 9) , then n cannot be represented as a sum of two squares.

iii) If n is the sum of two triangular numbers, then 4n+1 is the sum of two squares.

8) If the positive integer n is not the sum of squares of two integers, show that n cannot be represented as sum of two squares of rational numbers.

9) Let p be an odd prime. If p|a^2 + b^2, wheregcd (a , b)=1, prove that the prime p ≡ 1(mod 4).

10) Prove that a positive integer is representable as the difference of two squares if and only if it is the product of two factors that are both even or both odd.

11) Prove that a positive even integer can be written as the difference of two squares if and only if it is divisible by 4.

12) Prove that any positive integer can be written as the sum of four squares , some of which may be zero.

13) Let q be a prime factor of $a^2 + b^2$. If q ≡ 3(mod 4), then show that q|a and q|b.

14) Prove that every integer n ≥ 170 is a sum of five squares, none of which is equal to zero.

15) Show that the following equations have no solutions in integers.

i) $15x^2 - 7y^2=9$

ii) $y^2= x^3 + 7$

iii) $y^2 = 41x + 3$

iv) $x^2 + y^2 = 9z + 3$

……

Number Theory 5

Order of an integer & primitive roots, Cryptography

1) The order of integer 2 modulo 17 is
a) 16 b) 8 c) 17 d) None

2) If a has order n-1 then n is
a) Prime b) Composite c) Power of 2 d) None

3) The odd prime divisor of $n^2 + 1$ is of the form
a) 4k + 3 b)4k - 1 c) 4k + 1 d) None

4) The order of a is h modulo n then
a) h/Φ(n) b) Φ(n)/h c) h≠Φ(n) d) None

5) If a has order 8 modulo n then a^4 has order
a) 4 b) 2 c) 8 d) None

6) 2 is not primitive root of
a) 17 b) 19 c) 9 d) None

7) 3 is a primitive root of
a) 17 b) 19 c) 9 d) None

8) Number of primitive roots of 10 is
a) 4 b) 2 c) 5 d) None

9) With Caesar cipher, $f\ x = x + 3(mod\ 26)$, "YES" is enciphered as,
a) NO b) BHV c) XYZ d) YES

10) With Caesar cipher, encrypted message "ZKB" is deciphered as,
a) CNE b) BKZ c) WHY d) YES

11) The decryption function $f^{-1}$for the shift cipher $f\ x = x + 5(mod\ 26)$ is

a)$f^{-1}\ x = x + 21(mod\ 26)$ b) $f^{-1}\ x = x - 5(mod\ 26)$

c) $f^{-1}\ x = x - 21(mod\ 26)$ d) None

12) In an affine cryptosystem, $f\ x = 7x + 12(mod\ 26)$ "PAYMENOW" is encrypted as
a) NOWPAYME b) NMYSOZGK c) AYPNEMWO d) None

13) The decryption function $f^{-1}$for $f\ x = 7x + 12(mod\ 26)$ is given by

a)$f^{-1}\ x = 15x + 2(mod\ 26)$ b) $f^{-1}\ x = 12x + 7(mod\ 26)$

c) $f^{-1}\ x = (x/2) - 12(mod\ 26)$ d) None

14) The number of distinct shift cipher *mod26*is
a) 26 b) 25 c) 3 d) 676

15) The number of distinct shift cipher *mod26*is
a) 676 b) 26 c) 12 d) 312

Descriptive Questions

1) In a 27-letter alphabet (A = 0, B = 1, ……………., Z = 25, blank = 26) use affine encryption system $f(x) = ax + b(mod\,27)$ with a = 13, b = 9 to encipher the message "HELP ME". Also find the decryption function.

2) Encipher "JACK AND JILL" with
a) $f(x) = 5x + 8(mod\,26)$ (blanks are not taken into account)
b) $f(x) = 5x + 8(mod\,27)$ (blank = 26)

3) If VYKAR VAKEC is obtained using encryption function $f(x) = 17x + 10(mod\,26)$, then decipher it.

4) If $f(x) = ax + b(mod\,26)$ interchanges N & S, then find a, b.

5) If $f(x) = ax + b\ mod\,26$ leaves N fixed, then find possible values of b.

6) Encipher "WHATASURPRISE" twice with $f(x) = 3x + 2\ mod\,26$.

7) Find the number of distinct shift encryption systems given by $f(x) = x + a\ mod\,n$ with
a) n = 27
b) n = 29
c) n = 30

8) Find the number of distinct affine encryption systems given by $f(x) = ax + b\ mod\,n$ with
   a) n = 27
   b) n = 29
   c) n = 30

9) Find the number of distinct affine encryption systems given by $f(x) = ax + b\ mod\,n^2$ where x is a diagraph with
   a) n = 26
   b) n = 27
   c) n = 29
   d) n = 30

10) Find the order of the integers 2, 3 and 5.
   a) modulo 17
   b) modulo 19

c) modulo 23

11) Establish each of the statements below.

a) If a has an order hk modulo n, then a^h has order k modulo n.

b) If a has an order 2k modulo the odd prime p, then $a^k \equiv -1 \ (mod\, p)$.

c) If a has an order n-1 modulo n, then n is a prime.

12) Prove that $\Phi(2^n - 1)$ is multiple of n for any n > 1.

13) Assume that the order of n modulo h and the order of b modulo n is k. Show that the order of ab modulo n divides hk; in particular, if gcd (h , k) = 1, then ab has order of hk.

14) Given that a has order 3 modulo p where p is an odd prime show that a + 1 must have order 6 modulo p.

15) Verify the following assertion:

The odd prime divisors of integer $n^2 + 1$ are of the form 4k + 1.

16) Establish that there are infinitely many primes of each of the forms 4k + 1.

17) Prove that if p and q are odd primes and q| $a^p - 1$, then either q | $a - 1$ or else q = 2kp + 1 for some integer k.

18) a) Verify that 2 is a primitive root of 19, but not of 17.
b)Show that if 15 has no primitive root by calculating the orders of 2, 4, 7, 8, 11, 13 and 14 modulo 15.

19) Let r be a primitive root of the integer n. Prove that r^k is a primitive root of n if and only if gcd (k, Φ(n)) = 1.

20) Find 2 primitive roots of 10.

21) Assuming that r is a primitive root of odd prime p, establish the following facts:

a) The congruence $r^{(p-1)/2} \equiv -1 \ (mod\, p)$ holds.

b) If r' is any other primitive root of p then rr' is not a primitive root of p.

c) If the integer r' is such that rr' $\equiv 1 \ (mod\, p)$then r' is a primitive root of p.

22) Let r be a primitive root of the odd prime p, prove the following:

a) If p $\equiv 1 \ \ mod 4$, then -r also has a primitive root of p.

b) If p $\equiv 3 \ \ mod 4$ then -r has order (p-1)/2 modulo p.

23) For a prime p >3, prove that primitive roots of p occur in incongruent pairs r, r' where rr' $\equiv 1 \ \ mod\, p$.

24) a) Find the four primitive roots of 26 and the eight primitive roots of 25.

b) Determine all the primitive roots of $3^2, 3^3$ and 3^4.

25) For an odd prime p, establish the following facts:

a) There are as many primitive roots of $2p^n$ as that of p^n.

b) Any primitive root r of p^n is also a primitive root of p.

c) A primitive root of p^2 is also a primitive root of p^n for n ≥ 2.

Practical No. 6

Cryptography

Objective Questions

1) A digraph xy has value 26x+y Then “NO” has value

(a) 0 (b) 260 (c) 352 (d) none

2) For a diagraph x, if f(x)= 159x+580(mod26) is

(a) 26 (b) 676 (c) 26! (d) none

3) The matrix A= $\begin{matrix} 2 & 2 \\ 21 & 8 \end{matrix}$ $\in M_2(Z_{26})$ is

(a) Invertible (b) singular (c)non-singular (d) none

4) The matrixA= $\begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix}$ $\in M_2(Z_{26})$ is

(a) Invertible (b) singular (c)identity (d) none

5) The inverse of A= $\begin{matrix} 2 & 3 \\ 7 & 8 \end{matrix}$ $\in M_2(Z_{26})$ is

(a) $\begin{matrix} -2 & -3 \\ -7 & -8 \end{matrix}$ (b) $\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$ (c)) $\begin{matrix} 14 & 11 \\ 17 & 10 \end{matrix}$ (d) none

6) Let f(x)=Ax where A $\in M_2(Z_{26})$ and x is block of two letters, then for x=”NO” and A= $\begin{matrix} 2 & 3 \\ 7 & 8 \end{matrix}$ ciphertext f(x) is given by

(a) QV (b)YES (c)ON (d) none

7) Let f(x)=Ax where A $\in M_2(Z_{26})$ and x is block of two letters, then for f(x)=”FW” and A= $\begin{matrix} 2 & 3 \\ 7 & 8 \end{matrix}$, x is

(a)FW (b)WF (c)AT (d)NO

8) Let A,B$\in M_2(Z_{26})$ with invertible A and x be block of 2 letters .For f(x)=Ax+B the decryption Function f^{-1}(x) is given by

(a) Bx+A (b)Ax-B (c)A^{-1}x-A^{-1}B (d) none

9)The number of distinct Hill ciphers is

(a)157248 (b)26^4 (c)26! (d) none

10) Given n=19939,φ(n)=19656 .If n is product of primes p and q ,then p and q are

(a)151,129 (b) 157, 127 (c) 199 ,31 (d)none

11) If n=pq where p, q are distinct odd primes. if e= $\frac{\varphi(n)}{2}$ +1, then g. c. d. (e, $\varphi(n)$) is

(a) e (b) $\varphi(n)$ (c) 1 (d) 2

12) If f(x)≡ x^7(mod187) then f (3) is

(a) 130 (b) 3 (c) 21 (d) none

13) If (n,e)=(3233,37) is enciphering key in RSA cryptosystem then deciphering exponent is

(a) 253 (b) 273 (c) 453 (d) none

14) In ElGamal cryptosystem k=15 is secret key of public encryption key

(a)(37,2,18) (b) (113,3,24) (c)31,2,22) (d)none

15)Using Vigenereautokey cipher with seed Q message HAPPY is enciphered as follows

(a)XHPEN (b)HPXEN (c)PEXHN (d)none

Descriptive Questions

1) Find inverse of a matrix mod n, if it exists

a) A = $\begin{matrix} 1 & 3 \\ 4 & 3 \end{matrix}$ mod 5

b) A = $\begin{matrix} 1 & 3 \\ 4 & 3 \end{matrix}$ mod 29

c) A = $\begin{matrix} 15 & 17 \\ 4 & 9 \end{matrix}$ mod 26

d) A = $\begin{matrix} 40 & 0 \\ 0 & 21 \end{matrix}$ mod 841

2) Encipher "SEND" by taking 2 blocks SE & ND, first applying $\begin{matrix} 3 & 11 \\ 4 & 15 \end{matrix}$ mod 26& then $\begin{matrix} 10 & 15 \\ 5 & 9 \end{matrix}$ mod 29. Explain how to decipher it.

3) For any matrix $\begin{matrix} a & b \\ c & d \end{matrix}$ mod 26, show that "AA" is always fixed. Find condition such that it is the only fixed block.

4) Let n = pq, product of 2 distinct primes. If n = 63083 & Φ(n) = 62568, find p, q.

5)

A) Encipher the message HAVE A NICE TRIP using a Vigenere cipher with the keyword MATH.

B) The ciphertext BS FMX KFSGR JAPWL is known to have resulted from a Vigenere cipher whose keyword is YES. Obtain the deciphering congruences and read the message.

6) The message REPLY TODAY is to be encrypted in the ElGamal cryptosystem and forwarded to a user with public key (47,5,10) and private key k=19,if the random integer chosen for encryption is f=15, determine the ciphertext.

7) Suppose that the following ciphertext is received by a person having ElGamal public key (71,7,32) and private key k=30:

(56, 45) (56,38) (56,29) (56,03) (56,67)

(56,05) (56,27) (56,31) (56,38) (56,29)

Obtain the plain text message.

8)

A) Encipher the message HAPPY DAYS ARE HERE using the autokey cipher with seed Q.

B) Decipher the message BBOT XWBZ AWUVGK, which was produced by the autokey cipher with seed RX.

9) When the RSA algorithm is based on the key (n, k) = (3233,37), what is the recovery exponent for the cryptosystem?

10) Encipher "MATH" using RSA with n= 33 and e=5.

11) Decrypt the cipher text 00 41 46 04 01 43 00 that was encrypted with RSA algorithm with key (n, e) = (65, 7).

Miscellaneous Unit I

1) State & Prove Fermat's Theorem.
2) State & Prove Euler's generalization of Fermat's Theorem.
3) State & Prove Wilson's Theorem.
4) Let p be a prime. Show that $x^2 \equiv -1 \pmod p$ has solutions if and only if p=2 or $p \equiv 1 \pmod 4$
5) State & Prove Chinese Remainder Theorem.
6) Prove that $ax \equiv ay \pmod m$ iff $x \equiv y \left(\bmod \frac{m}{(a,m)}\right)$
7) P-T $x \equiv y$(modmi) for i=1 to n iff $x \equiv y \pmod{[m_1, m_2, \ldots, m_n]}$.
8) If $x \equiv y \pmod m$ them (x,m) = (y,m).
9) Let (a,m) =1 and $\{r_1, r_2, \ldots, r_{\phi(m)}\}$ be a reduced residue system mod m then $\{ ar_1, ar_2, \ldots, ar_{\phi(m)})$ is also a reduced residue system mod m.
10) Let m and n denote any two positive, relatively prime integers. Then $\phi(nm) = \phi(n)\, \phi(m)$.
11) If n> 1 then $\phi(n) = n \prod_{p/n}(1 - 1/p)$. Al so $\phi(1)=1$.
12) For $n \geq 1$ we have $\Sigma_{d/n}\, \phi(d) = n$.
13) The linear congruence $ax \equiv b \pmod n$ has a solution if and only if d/b where d =(a,n) .If d/b then it has d mutually incongruent solutions modulo n.
14) Explain the method for solving congruence $ax^2 + bx + c \equiv 0 \pmod p$ of degree 2 where p is an odd prime.
15) Let N(m) denote the number of solutions of the congruence $f(x) \equiv 0 \pmod m$. Then N(m) $= \prod_{i=1}^{r} N(pi^{ei})$ where m = $p1^{e1} .p2^{e2}$.............pr^{er} is the canonical factorization of m

MISCELLANEOUS QUESTIONS: Unit II

1)	Show that the linear Diophantine equation a x + b y =c has a solution if and only if d| c , where d= gcd (a, b). If x_0, y_0 is any particular solution of this equation , then all other solutions are given by $x = x_0 + \frac{b}{d}\, t$; $y = y_0 - \frac{a}{d}\, t$, where t∈ ▯
2)	If a and b are relatively prime positive integers, prove that the Diophantine equation ax-by=c has infinitely many solutions in positive integers.
3)	Let a, b, c be positive integers. Prove that there is no solution of ax+by=c in positive integers if a+b>c.
4)	Prove that a x + b y = a + c is solvable if and only if a x + b y = c is solvable.
5)	Show that all the solutions of the Pythagoras equation $x^2 + y^2 = z^2$

	satisfying the conditions gcd(x, y, z)=1, 2|x; x>0, y>0 ,z>0 are given by the formulae x= 2st y =$s^2 - t^2$ z =$s^2 + t^2$ for integers s>t>0 such that gcd (s, t)=1 and s $\not\equiv$ t(mod 2)
6)	Prove that the Diophantine equation$x^4 + y^4 = z^2$ has no solution in positive integers.
7)	Prove that the equation $x^4 + y^4 = z^4$ has no solution in positive integers.
8)	Prove that the Diophantine equation $x^4 - y^4 = z^2$ has no solution in positive integers.
9)	Prove that a positive integer n is representable as the sum of two squares if and only if each of it's prime factors of the form 4k+3 occurs to an even power.
10)	An odd prime p is expressable as a sum of two squares if and only if p≡1(mod 4).
11)	No positive integer of the form $4^n(8m + 7)$ can be represented as the sum of three squares.
12)	Any prime p can be written as sum of four squares.

Miscellaneous Questions : Unit III

1) Let n = pq, product of 2 distinct primes. If then prove $x^{ed} = x(mod\ n)$ with e and d, the enciphering & deciphering exponents of RSA system.

2) If e = $\frac{\Phi(n)}{2} + 1$ in RSA system, then prove for any *x*, $x^e \equiv x(mod\ n)$

3) In n is prime, then show that f(x) = ax + b(mod n) that $a \neq 1$, has a unique fixed point, i.e. f(x) = x.

4) Explain "shift" cryptosystem modulo n.

5)
A) Explain "affine" cryptosystem modulo n.
B) Find number of affine transformations ax + b(mod n) , for given $n \in N$.

6) If affine transformation is given by f(x) = ax + b(mod n) where b = 0 then prove f has at least 1 fixed point.

7) Explain hill Cipher with blocks of 2 letters.

8) Let A = $\begin{matrix} a & b \\ c & d \end{matrix}$ $\in M_2\ (Z_n)$.Prove, A is invertible iff g.c.d. (D, n) = 1 where D = detA modulo n

9) Explain RSA cryptosystem.

10)

A) Define order of an integer modulo n. If the integer a have order k modulo n prove that $a^n\ \equiv 1\ (modn)$iff k/h.

B) If the a has order k modulo n, prove that $a^i\ \equiv\ a^j(modn)$ iff $i \equiv j(modk)$.

C) If the integer a has order k modulo n & h > 0 then show that a^hhas order k/g.c.d.(h,k)modulo n.

11)

A) Define primitive root of integer n. If (a,n) = 1. If (a,n) = 1 & if $a_1, \dots\dots\dots a_{\Phi(n)}$ are positive integers less than n relatively prime to n. If a is primitive root of n then a,a^2,........$a^{\Phi(n)}$ are congruent modn to $a_1, \dots\dots\dots a_{\Phi(n)}$ in same order.

B) If a has primitive root. Show that it has exactly Φ(Φ(n)) primitive roots.

C) For k ≥ 3 prove that 2^k has no primitive.

D) If g.c.d. (m,n) = 1 where m > 2 & n > 2 then prove that the integer mn has no primitive roots.

E) If p is an odd prime, prove that there exists a primitive root of p such that $r^{p-1} \not\equiv$ 1(mod p^2).

F) Let p be an odd prime and r be a primitive root of p with the property that $r^{p-1} \not\equiv$ -1 (mod p^2) Prove that for each positive integer k ≥ 2, $r^{p^{k-2}}$ ≡ 1(mod p^k).

G) If p is an odd prime no. & k ≥ 1 then prove that there belongs a primitive root for p^k.

H) Prove that an integer n > 1 has a primitive root if and only if n = 2, 4,p^k or $2p^k$.

T.Y.B.Sc- Mathematics Semester VI (2018-19)

Paper 1 Practical 1

Limits, Continuity and Derivatives of functions of a Complex variable

Objective Questions

1. $\lim\limits_{Z\to 0} \frac{Im(z)}{z}$
   (a) 1 (b) i (c) $-i$ (d) does not exist
2. $\lim\limits_{Z\to 0} \frac{\bar{z}}{z}$
   (a) 1 (b) i (c) $-i$ (d) does not exist
3. If $f(z) = \begin{cases} \bar{z}^3 & if\ z \neq 0 \\ z & if\ z = 0 \end{cases}$ then
   (a) f is not continuously only at 0 (b) f is continuous on $\mathbb{C}$
   (c) f is discontinuous at 0 (d) None of the above
4. $f(z) = \frac{z^2+1}{z^3+9}$
   (a) Continuous and bounded in $|z| \leq 2$ (b) Continuous but not bounded in $|z| \leq 2$
   (c) Neither continuous nor bounded in $|z| \leq 2$ (d) Continuous and bounded everywhere
5. $f(z) = \frac{1}{z}$ is
   (a) Continuous and bounded in $|z| > 0$ (b) Continuous but not bounded in $|z| < 0$
   (c) Neither continuous nor bounded in $|z| > 0$ (d) Continuous and bounded everywhere
6. $\lim\limits_{x\to\infty} n\, i^n$ is
   (a) Does not exist (b) 1 (c) 0 (d) None of these
7. $\frac{(1-i)^{23}}{(\sqrt{3}-i)^{13}}$ in the polar from equals
   (a) $2^{-3/2}\ e^{5i\pi/12}$ (b) $e^{5i\,\pi/12}$ (c) $2^{-3/2}$ (d) None of these
8. $5e^{3\pi i/4} + 2e^{-\pi i/6}$ equals
   (a) $\left(\frac{-5\sqrt{2}+2\sqrt{3}}{2}\right) + i\left(\frac{5\sqrt{2}-2}{2}\right)$ (b) 0 (c) 1 (d) i
9. $z = a + ib,\ z^n = 1.$ Then z^{n-1} expressed in the form $A + iB$ is
   (a) $a^{n-1} + ib^{n-1}$ (b) $a^{n-1} - ib^{n-1}$ (c) 0 (d) None of these
10. $z_1 = 1/2 + i, z_2 = \sqrt{2} + i(\sqrt{2}+1), z_3 = 2 + 3i,\ \ z_4 = \frac{-1}{2} + i\sqrt{3}$, which of the points lie inside the circle $|z - i| = 2$
    (a) z_1, z_2, z_3 (b) z_1, z_2, z_4 (c) z_2, z_3, z_4 (d) None of these
11. Non- zero vectors z_1 & z_2 are perpendicular iff
    (a) $Re(z_1 z_2) = 0$ (b) $Re\ z_1 \times Im\ z_2 = 0$ (c) $Re(z_1\ \bar{z}_2) = 0$ (d) $Im(\overline{z_1}, z_2) = 0$

12. Let $|z| = 1$ or $|w| = 1.$ Then $|z - w| =$
    (a) $|1 - \bar{z}w|$ (b) $|1 - z\,w|$ (c) $|1 - \bar{z}\,\bar{w}|$ (d) None of these
13. (i) $\bar{z} = z$ (ii) $Re(z) = \frac{z+\bar{z}}{2}$ (iii) $Im(z) = \frac{z-\bar{z}}{2i}$ (iv) $Re\ (iz) = -Im(z)$ (v) $\text{Im}(iz) = R_e(z)$
    a) Only (i), (ii), (iii) are true. b) Only (iv) (v) are true
    c) All statements (i) (v) are true d) None of the above.

14. $(-\sqrt{3} - i)^{30} =$
    (a) 2^{30} (b) -2^{30} (c) $-2^{30} - i$ (d) $2^{30} + i$

15. $f(z) = 4x^2 + i\,4y^2 =$

(a) $z + \bar{z}$ (b) $z\,\bar{z}$ (c) $(1-i)z^2 + (2+2i)z\,\bar{z} + (1-i)\bar{z}^2$ (d) None of these

16. If an ellipse $s(t) = 2\,cost + i\sin t\,, 0 \le t \le 2z$ is rotated by $^{z}/_{6}$ and centre shifted to $2+i$, then parametric equation $r(t)$ of the resulting ellipse is

(a) $r(t) = \left(\sqrt{3}\,cost, \sqrt{2}\,sint\right)$

(b) $r(t) = \left(\sqrt{3}\,cost - \frac{1}{2}\sin t + 2, cost + \frac{\sqrt{3}}{2}\sin t + 1\right); 0 \le t \le 2\pi$

(c) $r(t) = \left(\sqrt{2}\,cost, \sqrt{3}sint\ \right)$

(d) None of these

17. The image of a circle under a linear transformation is

(a) straight line (b) circle (c) can be a straight line or a circle (d) any curve

18. $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree $n \ge 1$

Then for $k = 0$ to $n, a_k =$

(a) $k\,!p^k(0)$ (b) $\frac{P^k(0)}{k!}$ (c) $\frac{P^k(0)}{(K+1)!}$ (d) None of these

19. $\frac{d}{dz} z^n = n\mathbb{Z}^{n-1}$ is valid if

(a) $n \in \mathbb{Z},\ z \in \mathbb{C}$

(b) $n \in \mathbb{Z} \setminus \{0\}, z \in \mathbb{C}$

(c) $n \in \mathbb{N},\ z \in \mathbb{C} \setminus \{0\}$

(d) $n \in \mathbb{Z}\setminus\{0\}, z \in \mathbb{C}\setminus\{0\}$

20. $f'(z_0), g'(z_0)$ exists $g'(z_0) \ne 0$ $f(z_0) = 0 = g(z_0)$. Then $\lim_{z\to z_0} \frac{f(z)}{g(z)} =$

(a) does not exist (b) 0 (c) $\frac{f'(z_0)}{g'(z_0)}$ (d) $[f'(z_0)\ - g'(z_0)][g'(z_0)]^2$

21. $f(z) = {}^{1}/_{z}\ ,\ z \ne 0,\ f'(z) =$

(a) does not exist (b) $-\frac{1}{z^2}$ (c) 0 (d) None of these

22. $f(z) = Rez.\ f'(z)$ exists

(a) $\forall\, z \in \mathbb{C}$ (b) only at $z = 0$ (c) no where on $\mathbb{C}$ (d) exist only on real axis

23. $f(z) = Im\,Z,\ \ f'(z)$ exists

(a) $\forall\ z\ \in \mathbb{C}$ (b) only at $z = 0$ (c) no where on $\mathbb{C}$ (d) exists only on imaginary axix

24. $f(z) = z - \bar{z}\,, f'(z)$ exists

(a) only at 0 (b) only at i (c) on $\mathbb{C}$ (d) nowhere on $\mathbb{C}$

25. $f(z) = e^{-x}e^{-iy}$

(a) $f'(z)$ exists no where on $\mathbb{C}$ (b) $f'(z)$ exists on $\mathbb{C}$

(c) $f'(z)$ exists only at i (d) None of these

26. $f(z) = x^3 + i(1-y)^3$. Then

(a) f is differentiable only at $z = f'(z) = 3x^2$

(b) f is differentiable only at $z = i,\ \ f'(z) = 3z^2$

(c) f is differentiable only on $\mathbb{C}$ & $f'(z) = 3x^2 - i\,3(1-y)^2$

(d) f is differentiable only at $z = 0$ & $f'(z) = 0$

27. $f(z) = \begin{cases} \frac{\bar{z}^2}{z} & z \ne 0 \\ 0 & otherwise \end{cases}$

(a) Cauchy Riemann equations are not satisfied at (0,0)

(b) Cauchy Riemann equations are satisfied at (0,0) but f is not differentiable at (0,0)

(c) Cauchy Riemann equations are not satisfied at (0,0) but f is differentiable at $(0,0)$

(d) None of the above

28. $f(z) = Re(z), g(z) = Im\ (z), h(z) = \bar{z}$ Then which of the following statements hold

(a) f, g, h satisfies Cauchy Riemann equations at every pt

(b) f, g, h does not satisfies Cauchy Riemann equations at any pt.

(c) f, g satisfies Cauchy Riemann equations but h does not satisfy Cauchy Riemann equations.

(d) f, g does not satisfy Cauchy Riemann equations but h does satisfy Cauchy Riemann equations.

DESCRIPTIVE QUESTIONS

1) Use $\epsilon - \delta$ definition of limit to show that

$$\lim_{z\to 0}\frac{\bar{z}^2}{z} = 0 \quad \lim_{z\to 1-i}[x + i(2x+y)] = 1+i \quad \lim_{z\to 0}(\bar{z}^2+2) = 2i+2$$

2) Use $\epsilon - \delta$ definition of limit to show that that $\lim\limits_{z\to 0}\left(\frac{z}{z}\right)^2$ does not exist

3) Compute following limits

(i) $\lim\limits_{z\to 4i}\frac{z^2+16}{z-4i}$ (ii) $\lim\limits_{z\to i}\frac{z^4-1}{z^2-1}$ (iii) $\lim\limits_{z\to i}\frac{iz^3-1}{z+i}$

4) Show that

(i) $\lim\limits_{z\to\infty}\frac{4z^2}{(z-1)^2} = 4$ (ii) $\lim\limits_{z\to\infty}\frac{z^2+1}{z-1} = \infty$ (iii) $\lim\limits_{z\to 1}\frac{1}{(z-1)^3} = \infty$

5) Test for the continuity of the function

(i) $f(z) = \begin{cases}\frac{z^2+9}{z-3i} & if\ z \neq 3i\\ z-3i & if\ z = 3i\end{cases}$

(ii) $f(z) = \begin{cases}\frac{\bar{z}^3}{zRez} & if\ z \neq 0\\ 0 & if\ z = 0\end{cases}$

6) Represent the following subsets of $\mathbb{C}$ in the plane

(a) $|z-1+3i| = 2,\ \ |z+2| = |z-1|,$

$|z-z_0| = |z-\bar{z}_0|$ where $Im\ z_0 \neq 0,$

$|z-z_0| = |z+\bar{z}_0|$ where $Re\ \ z_0 \neq 0$,

$|z-2| = 2|z-2i|, \left|\frac{z-z_0}{z-z_1}\right| = c, c \neq 1, z_0 \neq z_1\ 0 < Im\ z < 2\pi,\ \frac{Re\ z}{|z-1|} > 1\ , Im\ z < 3$

(b) $|z+1-2i| = 2,\ Re(z+1) = 0,\ |z = 2i| \leq 1,\ Im(z-2i) > 6$

(c) $Re(z) \geq 2, Re\ (z^2) \leq \propto,\ Im(z^2) \leq \propto,$

$|z^2-2| \leq 1, \qquad \left|\frac{1}{z}\right| < 1, \qquad \left|\frac{z-1}{z+1}\right| \leq 1$

(d) $|z+1| - |z-1| = \pm 2$

7) $f\colon \Omega \subseteq \mathbb{C} \to \mathbb{C}$ Define differentiability (or complex differentiability) of f at $z_0 \in \Omega$

Using the definition above, discuss differentiability of the following function f at the point s mentioned.

(a) $f(z) = z^2$ for any $z \in \mathbb{C}$ (b) $f(z) = \bar{z}$ for any $z \in \mathbb{C}$

(c) $f(z) = |z|^2$ for any $z \in \mathbb{C}$ (d) $f(z) = \begin{cases}\bar{z}^2/z & z \neq 0\\ 0 & z = 0\end{cases}$ $at\ (0,0)$

8) Write the function $f(z) = |z|$ in the form $u(x,y) + i\ v(x,y)$ .Using Cauchy Riemann equations, decide whether they are any points in $\mathbb{C}$ at which f is differentiable

9) Use (i) Definition of differentiability

(ii) Cauchy Riemann equations

to check differentiability of $f(z) = Re\ \ z, f(z) = Im\ z$

10) Test differentiability of the following function at $(0,0)$. $f(z) = z\ Re\ z, f(z) = z\ Im\ z, f(z) = z|z|$

11) Use polar co-ordinates to show that $f(z) = |z|^2$ is complex differentiable at 0. what can you say about $f(z) = |z|$? Justify your answer.

12) Show that $f(z) = |z|$ is differentiable everywhere except at $z = 0$, when f is considered as a function from $\mathbb{R}^2 \to \mathbb{R}^2$. Is f $\mathbb{C}$ differentiable? Justify your answer.

13) Show that $f(z) = z\,|z|$ is differentiable everywhere when f is treated as a function from $\mathbb{R}^2 \to \mathbb{R}^2$ but $\mathbb{C}$ differentiable only at $z = 0$.

14) Show that $f(x, y) = \sqrt{|xy|}$ satisfies Cauchy Riemann equations at $(0, 0)$ but f is not $\mathbb{C}$ differentiable at $(0,0)$.

15) $f(z) = f(x, y) = \begin{cases} \left(\frac{xy}{x^2+y^2}, 0\right) & for\ (x, y) \neq (0, 0) \\ (0, 0) & otherwise \end{cases}$

Show that f satisfies Cauchy Riemann equation at $(0, 0)$ but not $\mathbb{C}$- differentiable at $(0, 0)$

16) $f(z) = \bar{z}e^{-|z|^2}$. Determine the points at which $f'(z)$ exist and find $f'(z)$ at these points.

17) f is $\mathbb{C}$-differentiable on an open disk such that its image id contained in a line, a circle, a parabola or a hyperbola. Show that f is a constant.

18) (a) $f(z) = z^3$. f is differentiable at $z_1 = 1$ and $z_2 = i$. Show that there does not exist a point c on the line $y = 1 - x$ between 1 & i such that $\frac{f(z_2)-f(z_1)}{z_2-z_1} = f'(c)$

(b) Does mean Value Theorem for derivatives of real valued functions hold for complex functions? Justify your answer.

19) Does mean value theorem for integrals hold for complex valued functions justify?
(Consider the function $w : [0, 2\pi] \to \mathbb{C}, w(t) = e^{it}$ Note- can this function also be used to show Mean value theorem is not true for derivatives for complex valued functions.)

PRACTICAL NO. 2

Stereographic Projection, Analytic functions, Finding Harmonic Conjugates

(1) $F(z) = x^3 + 3xy^2 + i(y^3 + 3x^2y)$ is analytic

a) only at $1, i$ b) only at 0 c) only at $0, 1, i$ d) nowhere on *

(2) $f(z) = (2x - y) + i(Ax + By)$ is an entire function then

a) $A =$ $, B =$ b) c) d)

(3) $f(z) = e^y \cos x + i\, e^y \sin x,\ \ g(z) = z + \bar{z}$. Then

a) Both f, g are analytic on $\mathbb{C}$ b) f analytic on $\mathbb{C}$ but g is not analytic on $\mathbb{C}$

c) f not analytic on $\mathbb{C}$ but g is not analytic on $\mathbb{C}$ d) Both f, g are not analytic on $\mathbb{C}$

(4) $f(z) = (z^2 - 2)e^{-x}\, e^{-iy}, g(z) = xy + iy$

$h(z) = 2xy + i(x^2 - y^2)$

a) f is an entire function, g and h are no where analytic

b) g is an entire function, f and h are no where analytic

c) h is an entire function, f and g are no where analytic

d) f, g, h all of them are analytic on $\mathbb{C}$

(5) $f(z) = x^3 + 3xy^2 + i(y^3 + 3x^2y)$ is

a) an entire function b) analytic on the unit disk

c) differentiable an x-axis d) differentiable on x & y axes but analytic nowhere

(6) The singular point of $f(z) = \frac{2z+1}{z(z^2+1)}$ are

a) only at 0 b) $0, \pm i$ c) only at $\pm\ i$ d) None of these

(7) $f(x + i\, y) = x^3 - 3xy^2 + i(3x^2y - y^3)$

a) f is analytic on $\mathbb{C}$ b) f is analytic only on the unit disk

c) f is analytic only on $\mathbb{C} \setminus \{0\}$ c) None of these

(8) The singular point of $f(z) = \frac{z^2+1}{(z+2)(z^2+2z+2)}$ are

a) $\pm\ i$ b) $-2, -1 \pm\ i$ c) 0 d) None of these

(9) $u(x, y) = x^2 - y^2, v = 2xy$

a) v & u are harmonic conjugates of each other

b) u is a harmonic conjugate of $\frac{v}{u}$ but v is not

c) v is a harmonic conjugate of $\frac{u}{v}$ but u is not

d) None of these

(10) $u = ax^3 + bxy$. For u to be harmonic, the value of a and b are

a) a= b= b) c) d)

(11) $f(z) = \frac{1}{z}, z \neq 0$. level sets of level sets of f real and imaginary parts of f are

a) Not orthogonal b) orthogonal c) equal c) None of these

(12) The image of a line under a fractional linear transformation is

a) a line b) a circle c) a line or a circle d) None of these

(13) The image of a circle under a Mobius transformation is

a) a point b) a line c) a circle d) a line or a circle

(14) $f(z) = i\,\frac{1-z}{1+z}$. The image of the unit circle under is

a) b) unit circle c) the imaginary axis d) the real axis

DISCRIPTIVE QUESTION

(1) Determine where the following functions are differentiable and where they are analytic

a) $f(z) = x^3 + 3xy^2 + i(y^3 + 3x^2y)$

b) $f(z) = 8x - x^3 - xy^2 + i(x^2y + y^3 - 8y)$

c) $f(z) = x^2 - y^2 + i\;2|xy|$

(2) Does there exist a complex differentiable function $f = u + iv$ with real part $u(x, y) = xe^y$? Justify your answer.

(3) Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ when

a) $u(x, y) = 2x(1 - y)$ b) $u(x, y) = x^3 - 3xy^2$

c) $u(x, y) = xy^3 - x^3y$ d) $u(x, y) = 2x - x^3 + 3y^2$

e) $u(x, y) = \sin h\; x\; \sin y$ f) $u(x, y) = \frac{y}{x^2+y^2}$

(4) Show that if u & v are harmonic conjugates of $u(x, y)$ in a domain D then $v(x, y)$ & $V(x, y)$ can different at most by an addictive constant.

(5) Prove the following functions harmonic? If so, function is corresponding analytic function

$f(z) = u(x, y) + i\;v(x, y)$ where

a) $u - x^3 + y^2$ b) $v = e^x \sin 2y$ c) $v = (2x + 1)y$

6) Describe stereographic projection and show that it is given by the $map \ \ \sigma :$
$S^2 \setminus \{(0,0,1)\} \to \mathbb{C}$ $\qquad \sigma(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}$

(6) Interpret the transformation

$f\colon \mathbb{C} \to \mathbb{C},\ f(z) = \left(1 + \sqrt{3}i\right)z$ Geometrically

(7) Show that $z(t) = z_0 + tv$ and $Re\left((z - z_0)i\ \bar{v}\right) = 0$ represents the same line in $\mathbb{C}$

T.Y.B.Sc- Mathematics-- Semester VI (2018-19)

Paper 1 Practical 3

Contour Integral, Cauchy Integral formula, Mobius transformation

Objective Questions

(1) The value of integral $\int_C \frac{e^z}{z-2}$, where C is the circle $|z| = 3$, described in positive sense is ____________

a) $2\pi i\ e^2$ (b) $2\pi i$ (c) e^2 (d) None of these

(2) If f is analytic in a simply connected domain D, then for every closed path C in D we have____________

(a) $\int_C f(z)dz = 1$ (b) $\int_C f(z)dz = 0$ (c) $\int_C f(z)dz \neq 0$ (d) None of these

(3) The value of integral $\int_C \frac{z}{2z+1} dz$, where C is the circle $|z| = 1$, described in positive sense is ____________

(a) $\frac{\pi i}{2}$ (b) $\frac{-\pi i}{2}$ (c) 0 (d) None of these

(4) Let C be the circle centered at 0 and radius 3 traversed once in the anti-clockwise sense, then value of the integral $\int_C \frac{e^2 z}{(z+1)^4} dz$ is ____________

(a) $\frac{4\pi i}{3}e^{-2}$ (b) $\frac{8\pi i}{3}e^{-2}$ (c) $\frac{-4\pi i}{3}e^{-2}$ (d) None of these

(5) The value of the integral $\int_\Gamma z^2 dz$, where $\Gamma = \Gamma_1 \cup \Gamma_2$ is given by $\Gamma_1 : \gamma_1(t) = e^{it}, 0 \leq t \leq \pi$ and $\Gamma_2 : \gamma_1(t) = e^{-it}, 0 \leq t \leq \pi$ is ____________

(a) $\frac{-4}{3}$ (b) $\frac{4}{3}$ (c) 4 (d) None of these

(6) Let C denotes the positively oriented boundary of the square whose sides lie on the lines $x = \pm 2$ and $y = \pm 2$, then $\int_C \frac{zdz}{2z+1}$ equals____________

(a) $\frac{-\pi i}{2}$ (b) $\frac{-\pi}{2}$ (c) $-2\pi i$ (d) None of these

(7) The value of integral $\int_C \frac{dz}{z^3(z+4)}$, where C is the circle $|z+2| = 3$, taken in counterclockwise direction equals____________

(a) $2\pi i$ (b) 0 (c) πi (d) None of these

(8) If $f(z)$ is analytic in domain D, then____________

(a) $f^{(n)}(z)$ exists in D (b) $f^{(n)}(z)$ does not exists in D

(c) $f^{(n)}(z) = 0,\ \forall\, n \in \mathbb{N}$ in D (d) None of these

Pract 3 objective (contd)

9. $I = \int_C \bar{z}\,dz$ where C is the line segment joining $-i$ to i. Then $I =$

a) does not exist (b) 0 (c) 1 (d) $1/2$

10. $f: \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}^2$. $\int_C f$ where C is the line segment from 0 to $1+i$ is

a) does not exist b) i c) $2/3(1-i)$ d) None of these

11. $I = \int_\gamma \bar{z}^2\,dz$ where $\gamma: |z-1| = 1$. Then $I =$

a) does not exist (b) $4\pi i$ (c) i (d) None of these

12. $I_n = \int_C (z-a)^n\,dz$, $n \in \mathbb{Z}$. $C: |z-a| = r$ with $a \in \mathbb{C}$, $r > 0$. Then $I_n =$

a) 0 $\forall n$ ~~exist~~ (b) $2\pi i$ $\forall n$ (c) $I_n = 0$ for $n \neq -1$, $I_{-1} = 2\pi i$ (d) $I_{-1} = 0$, $I_n = 2\pi i$ $\forall n \neq -1$

13. $\int_C \sin z\,dz = I$, where C is any Jordan curve in the complex plane. Then $I =$

a) π b) $2\pi i$ c) 0 d) None of these

14. $\int_C \frac{1}{(z^2+4)(z^2+9)}\,dz$ where $C: |z| = 1$ equals

a) $2\pi i$ (b) 1 (c) 0 (d) none of these

15. $I = \int_C \frac{z+1}{z^2(z-1)}\,dz$ where $C: |z-2| = \sqrt{2}$ traversed counterclockwise

a) $4\pi i$ (b) 0 (c) 1 (d) π

16. $\int_C \frac{z-1}{z^2(z-2)}\,dz$, $C: |z| = 1$ equals (a) π (b) $-\pi$ (c) $-\pi i/2$ (d) None of these

17. $\int_C \frac{\cos z}{z(z^2+8)}\,dz$, C bdd by $x = \pm 2$, $y = \pm 2$ equals a) b) c) d)

Descriptive Questions

(1) Evaluate $\int_0^{1+i}(x-y+ix^2)dz$

i) Along the line from $z=0$ to $z=1+i$

ii) Along the real axis from $z=0$ to $z=1$ then along the line parallel to the imaginary axis from $z=1$ to $z=1+i$

iii) Along the imaginary $z=0$ to $z=i$ then from $z=i$ to $z=1+i$

iv) Along the parabola $y^2=x$.

(2) Evaluate $\int_C \frac{1}{(z-z_0)^{n+1}}dz$ where C is the circle $|z-z_0|=r$

(3) Evaluate $\int_C \frac{2z+3}{z}\,dz$, where C is the

i) Upper half of the circle $|z|=2$

ii) Lower half of the circle $|z|=2$

iii) The whole circle $|z|=2$ in anti-clockwise direction

(4) Evaluate following.

i) $\int_C \frac{z+3}{z^2-2z+5}\,dz$, where $C:\ |z-1|=1$

ii) $\int_C \frac{\cos z}{z}\,dz$, where $C:\ 9x^2+4y^2=1$

iii) $\int_C \frac{1}{(z^3-1)^2}\,dz$, where $C:\ |z-1|=1$

iv) $\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{2}\right)^3}\,dz$, where $C:\ |z|=1$

v) $\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{2}\right)^3}\,dz$, where $C:\ |z|=2$

vi) $\int_C \frac{\exp(z)}{z^2-2z}\,dz$, $C: |z|=3$

vii) $\int_C \frac{dz}{z^2+4}\ dz\quad C:\ |z|=1$

viii) $\int_C \sec z\,dz\quad C:\ |z|=1$

ix) $\int_C \bar{z}\,dz\quad C:\ |z|=1$

x) $\int_C \frac{e^z}{z-2}\,dz$ where C is any simple, closed curve.

xi) $\int_C \frac{z^3-6}{2z-i}$ where C is any simple, closed curve.

xii) $\int_C \frac{z^2+1}{z^2-1}$ where (i) $C:\ |z-1|=1$ (ii) $C:\ |z+1|=1$ (iii) $C:\ |z-1|=1$

(5) Let γ " $[0,\pi] \to \mathbb{C}$ with $\gamma(t) = 2e^{it}$ be the positively semicircle in the upper half plane with center at the origin and radius 2. Prove that

$$\left|\int_\gamma \frac{e^z}{z^2+1}\,dz\right| \le \frac{2\pi e^2}{3}$$

(Do not try to evaluate the integral exactly, use ML inequality)

6) Find the image of the given set under the reciprocal map $w = \frac{1}{z}$ on the extended complex plane

(a) $|z| = 8$

(b) $|z| = 6, \quad {-\pi}/_6 \le \arg(z) \le {3\pi}/_4$

(c) $|z| = \frac{1}{4}, \quad {\pi}/_2 \le \arg(z) \le \pi$

(d) $\frac{1}{5} \le |z| \le 2$

(e) $\arg(z) = {\pi}/_3$

(f) $x = \frac{1}{5}\ f\ x = 2$

(g) $|z-2| = 2$

7 (a) Construal a linear fractional transformation that maps 0 to -1 i to 0 & ∞ to 1 respectively.

(b) Construal a linear fractional transformation that maps the points $i, \infty, 3$ to $\frac{1}{2}, -1, 3$ respectively.

8) Determine whether each of the following sets of points lies on a circle

(a) $0, -4, -2i, -1-3\,i$

(b) $-1, -i, i, 2-i$

(9) Evaluate $\displaystyle\int_C \frac{\sin h z\, dz}{(z-\pi i)^4}$ where $C: |z-2i|=3$, & positively oriented

T.Y.B.Sc- Mathematics– Semester VI

Paper 1 Practical 4

Taylor's series, Exponential, Trigonometric, Hyperbolic functions

Objective Questions

(1) If $f(z)$ is analytic at z_0 then following is the Taylor series of f at z_0,
($f^{(k)}$ *represents* k^{th} *derivative of* f.)
(a) $\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}z_0^k$ (b) $\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$ (c) $\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z+z_0)^k$ (d) None of these

(2) $f^{(3)}(0)$ for $f(z)=\sum_{n=0}^{\infty}(3+(-1)^n)^n z^n$ is ____________
(a) 2 (b) 8 (c) -2 (d) None of these

(3) $g^{(3)}(0)$ for $g(z)=\sum_{n=0}^{\infty}\frac{(1+i)^n}{n}z^n$ is
(a) $\frac{2}{3}(i-1)$ (b) $4(i-1)$ (c) $\frac{2}{3}(1-i)$ (d) None of these

(4) If $f(z)$ is analytic in domain D, then__________
(a) $f^{(n)}(z)$ exists in D (b) $f^{(n)}(z)$ does not exists in D
(c) $f^{(n)}(z)=0,\ \forall\, n\in\mathbb{N}$ in D (d) None of these

(5) $1-z+z^2$ - - - — - for $|z|<1$ is
a) $\frac{1}{1-z}$ b) $\frac{1}{i-z}$ c) $\frac{1}{i+z}$ d) $\frac{1}{1+z}$

(6) for $|z|<1$, Power series representation of $\frac{1}{(1-z)^2}$ is
a) $1+2z+3z^2+$ - - (b) $\sum_{n=0}^{\infty}z^n$ (c) $\sum_{n=0}^{\infty}(-1)^n z^n$ (d) None of these

(7) for $|z|<1$, power series representation of $\frac{1}{(1+z)^2}$ is
a) $1+2z+3z^2+$ - - (b) $1-2z+3z^2-$ - - - (c) $\sum_{n=0}^{\infty}(-1)^n z^n$ (d) Does not exist in ~~this~~ the given region.

(8) The Taylor's expansion of $f(z)=z^8e^{3z}$ around $z=0$ is
a) Does not exist (b) $\sum_{n=0}^{\infty}\frac{3^n}{n!}z^{n+8}$ (c) $\sum_{n=0}^{\infty}\frac{8^n}{n!}z^{n+3}$ (d) None of these

(9) $\exp(2 \pm 3\pi i) =$

a) e^{-2} b) $-e^{2}$ c) e^{2} d) e^{3}

(10) $\exp(2+\pi i/4) =$

a) $\sqrt{\frac{e}{4}}(2+i)$ b) $\sqrt{\frac{e}{4}}(4+\pi)$ c) $\sqrt{\frac{e}{2}}(1+i)$ d) $\sqrt{\frac{e}{4}}(4-\pi)$

(11) $\exp(z+\pi i) =$

a) $\exp z$ b) ~~exp~~ 0 c) $-\exp(z)$ d) None of these

12) $|\exp(-2z)| < 1$ iff

a) $\mathrm{Re}\, z > 0$ b) $\mathrm{Re}\, z \geq 0$ c) $\mathrm{Im}\, z < 1$ d) None of these

13) If $e^{z} = 2$, then $z =$

a) 0 b) 1 c) i d) $z = \ln 2 + (2n+1)\pi i$, $n = 0, \pm 1, \pm 2, --$

14) If e^{z} is real, then $\mathrm{Im}\, z =$

a) $n\pi$, $n = 0, \pm 1, \pm 2, --$ (b) $+n\pi/2$, $n \in \mathbb{N}$ c) i d) $-i$

15) $\overline{\exp(iz)} = \exp(i\bar{z})$ iff

a) $z = n\frac{\pi}{2}$, $n = 0, \pm 1, ---$ (b) $z = n\pi$, $n = 0, \pm 1, \pm 2, --$

c) $z = i$ (d) None of these

16) $f(z) = \sin \bar{z}$, $g(z) = \cos \bar{z}$

a) f is analytic on $\mathbb{C}$ but g is not analytic anywhere on $\mathbb{C}$

b) g is analytic on $\mathbb{C}$ but f is not analytic anywhere on $\mathbb{C}$

c) f & g are both analytic on $\mathbb{C}$.

d) Neither f nor g is analytic anywhere on $\mathbb{C}$

17) $\overline{\cos(iz)} = \cos(i\bar{z})$ for

a) $z = n\pi i$, $n \in \mathbb{N}$ (b) $\forall\, z \in \mathbb{C}$ (c) for $z = 0$ (d) None of these

18) $\overline{\sin(iz)} = \sin(i\bar{z})$ iff

a) $z = n\pi i$, $n \in \mathbb{Z}$ (b) $\forall\, z \in \mathbb{C}$ (c) for $z = 0$ (d) None of these

19) Roots of the equation $\sinh z = i$ are

a) $z = (2n+\frac{1}{2})\pi i$, $n \in \mathbb{Z}$, (b) $z = (n+1)\pi i$, $n \in \mathbb{Z}$

(c) $z = n\pi/2$, $n \in \mathbb{Z}$ (d) None of these

20 Roots of the equation $\cosh z = 1/2$ are

a) $(2n+1)\pi i, n \in \mathbb{Z}$ b) $z = \left(2n \pm \frac{1}{3}\right)\pi i, n \in \mathbb{Z}$

c) $(2n+1)\frac{\pi}{2} i, n \in \mathbb{Z}$ (d) $\left(2n - \frac{1}{3}\right)\pi i, n \in \mathbb{Z}$

Descriptive Questions

1) Establish the following properties of exponential function : e^z.
   (a) $e^{z1+z2} = e^{z1} * e^{z2}$ (b) $e^{z1-z2} = e^{z1}/e^{z2}$
2) Prove that $e^z \neq 0 \ \forall z \in \mathbb{C}$
3) Prove that e^z is periodic function with period $2\pi i$
4) Show that
   (a) $\exp(2 \pm 3i\pi) = -e^2$ (b)$\exp\left(\frac{2+\pi i}{4}\right) = \sqrt{\frac{e}{2}}(1+i)$ (c) $\exp(2+\pi i) = -\exp z.$
5) Write $|exp(2z+i)|$ and $|exp(iz^2)|$ in terms of x & y. Then show that $|exp(2z+1) + exp(iz^2)| \leq e^{2x} + e^{-2xy}$.
6) Prove that $|exp(-2z) <|$ iff $Re\ z > 0$
7) Find all values of z such that
   (a) $e^z = -2$ (b) $e^z = 1 + \sqrt{3}\,i$ (c) $\exp(2z-1) = 1$
8) Prove the following identities
   (a) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.
   (b) $\cos(z_1 + z_2) = \cos z_1\ cos\ z_2 - \sin z_1 \sin z_2$
   (c) $sin^2 z + cos^2 z = 1$
   (d) $\sin 2z = 2 \sin z \cos z$
   (e) $\cos 2z = cos^2 z - sin^2 z$
   (f) z
   (g) $1 + tan^2 z = sec^2 z$
   (h) $1 + cot^2 z = cosec^2 z$
9) Show that (a) $\overline{\cos(iz)} = \cos(i\,\bar{z})\ \forall\ z \in \mathbb{C}$
10) $\overline{\sin(iz)} = \sin(i\,\bar{z})$ iff $z = n\pi\, i\ (n = 0, \pm 1, \pm 2, \ldots)$
11) Find all roots of the equation $\sin z = \cosh 4$
12) Find all roots of the equation $\cos z = 2$.
13) Give definition of hyperbolic $\sin z$ ($i.e \sin hz$) & hyperbolic $\cos z$ ($i.e \cos h\ z$)
14) Prove that $\frac{d}{dz}\sin hz = \cos h\ z$ & $\frac{d}{dz}\cos h\ z = \sin h\ z$
15) Prove the following identities about $\sin h\ z$ & $\cos h\ z$, where z = x+iy
   (a) $\cos h^2 z - \sin h\ \ ^2 z = 1$
   (b) $\sin h(z_1 + z_2) = \sin h\ z_1 \cos h\ z_2 + \cos h\ z_1 \sin h\ z_2$
   (c) $\cos h\ (z_1 + z_2) = \cos h\ z_1 \cos h\ z_2 + \sin h\ z_1 \sin h\ z_2$
   (d) $\sin h\ z = \sin h \cos y + i \cos h\ x \sin y$
   (e) $\cos h\ z = \cos hz \cos y + i \sin h\ x \sin y.$
   (f) $|\sin h\ z|^2 = \sin h^2 x + sin^2 y$
   (g) $|\cos h\ z|^2 = \sin h^2 x + cos^2 y$
16) Show that (a) $\sin h(z + \pi i) = -\sin h\ z$ (b) $\cos h\ (z + \pi i) = -\cos h\ z$ (c) $\tan h\ (z + \pi\, i) = \tan hz$

17) Find all terms of $\sin h\ z$ & $cas\ h\ z$. Justify your answer

18) Find all roots of the equation

(a) $\sin h\ z = i$ (b) $\cos h\ z = 1/2$ (c) $\cos h\ z = -2$

19)Find Taylor series expansion of following $f(z)$ at $z = 0$

i) e^z ii) $\sin z$ iii) $\cos z$

(20)Expand $f(z) = \frac{1}{1-z}$ in a Taylor series with centre $z_0 = 2i$.

(21) Expand $f(z) = \frac{e^z}{1-z}$ in a Taylor series around z=0.

T.Y.B.Sc- Mathematics – Semester VI

Paper 1

Unit-I

1. If z_0 and w_0 are points in z and w plans respectively then show that

   (a) $\lim_{z\to z_0} f(z) = \infty$ if and only if $\lim_{z\to z_0} \frac{1}{f(z)} = 0$.

   (b) $\lim_{z\to\infty} f(z) = \omega_0$ if and only if $\lim_{z\to 0} f\left(\frac{1}{z}\right) = \omega_0$

   (c) $\lim_{z\to\infty} f(z) = \infty$ if and only if $\lim_{z\to 0} \frac{1}{f(1/z)} = 0$.
2. Suppose $w = f(z)$ is continuous at z_0 and $z = g(\zeta)$ is continuous at ζ_0. If $z_0 = g(\zeta_0)$, then show that function fog is continuous at ζ_0.
3. If a function $f: \Omega \to \mathbb{C}$ is continuous at $z_0 \in \Omega$ and $f(z_0) \neq 0$ then show that $f(z) \neq 0$ throughout some neighbourhood of z_0.
4. If a function f is continuous throughout region R that is closed and bounded then show that exist a non-negative integer M such that $|f(z)| \leq M \quad \forall z \in R$.

5. $f: A \subseteq \mathbb{C} \to \mathbb{C}$ is differentiable at $z_0 \in A$. Show that f is continuous at z_0.

6. Using definition of differentiability, show that if $f^{'}(z_0)$, $g'\big(f(z_0)\big)$ exist then prove that the function

   $F(z) = g\big(f(z)\big)$ has a derivative at z_0 and $F'(z_0) = g'\big(f(z_0)\big)f'(z_0)$.

7. Let $f: \Omega \subset \mathbb{C} \to \mathbb{C}$ such that f is differentiable at $z_0 \in \Omega$. Show that $\exists$ a function $\eta(z)$ such that

   $f(z) = f(z_0) + f^{'}(z_0)(z - z_0) + \eta(z)(z - z_0)$ where $\eta(z) \to 0$ as $z \to z_0$.

8. $f(z) = u(x, y) + iv(x, y)$. $f^{'}(z)$ exists at a point $z_0 = x_0 + iy_0$. Then prove that the first order partial derivatives of u & v exist at (x_0, y_0) and they satisfy Cauchy- Riemann equations $u_x = v_y$, $u_y = -v_x$. Also show that $f^{'}(z) = (u_x)_{z=z_0} + i(v_x)_{z=z_0}$. Show that the converse is not true.

9. $\Omega \subset \mathbb{C}$ is a domain. If $u, v : \Omega \to \mathbb{R}$ are such that

   i) u_x, u_y, v_x, v_y exist and satisfy Cauchy Riemann equations

   ii) u_x, u_y, v_x, v_y are continuous on Ω,

   then prove that $f(z) = u(x, y) + iv(x, y)$ is analytic in Ω.

10. If $f'(z) = 0$ everywhere on a domain D then show that $f(z)$ must be constant through out D.
11. Suppose that function $f(z)$ and $\overline{f(z)}$ are both analytic in a given domain D then show that $f(z)$ must be constant throughout D.
12. f is analytic throughout on a given domain D. If $|f(z)|$ is constant on D, show that $f(z)$ must be constant on D.

13. If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then show that its component function u and v are harmonic in D.

14. Show that $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.
15. Suppose that v is a harmonic conjugate of u in a domain D and also that u is a harmonic conjugate of v in D. Show that both $u(x, y)$ and $v(x, y)$ must be constant through out D.
16. Show that v is a harmonic conjugate of u in a domain D, iff $-u$ is a harmonic conjugate of v in D.
17. Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D and consider the family of level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ where $c_1, c_2 \in \mathbb{R}$. Prove that these families are orthogonal.

Unit-II

1. State and Prove Cauchy Goursat theorem. (weaker form ie with the hypothesis of f'(z) being continuous) (along with a problem)
2. Let A be an open connected subset of $\mathbb{C}$ and $f: A \to \mathbb{C}$ be an analytic function in A. Let $z_0 \in A$ and $r > 0$ such that $B(z_0, r) \subset A$. Then for any $\omega \in (z_0, r)$ prove that $f(\omega) = \frac{1}{2\pi i} \int_{\partial B(z_0, r)} \frac{f(z)}{z-\omega} dz$. (Cauchy Integral Theorem)
3. State and Prove extension of Cauchy's Integral formula. f is a analytic inside and on a simple, closed curve C, taken in the positive sense. Prove that $f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)ds}{(s-z)^2}$. Further state the result generalizing the formula to $f^n(z)$.
4. State and prove Taylor's theorem.
5. Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$, centered at z_0 and with radius R_0. Then prove that $f(z)$ has the power series representation $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, $|z - z_0| < R_0$ where $a_n = \frac{f^n(z_0)}{n!}$ ie the series converges to $f(z)$ when z lies in the stated open disk. (Taylor's theorem).
6. $w(t): [a, b] \to \mathbb{C}$ is a piecewise continuous function, then show that $\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)dt|$. Use this to prove ML Inequality.
7. Let C denote a contour of length L and suppose that a function $f(z)$ is piecewise continuous on C. If M is a non negative constant such that $|f(z)| \leq M \ \forall z \in C$ at which $f(z)$ is defined then $\left|\int_C f(z)dz\right| \leq ML$.

(ML Inequality)

8. State and Prove Cauchy Goursat theorem. (weaker form ie with the hypothesis of f'(z) being continuous)
9. State Cauchy Integral formula (extension). Hence, prove that

(i) If a function f is analytic at a given point then its derivatives of all orders are analytic at that point too.

(ii) If a function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point $z = (x, y)$, then the componenet functions u and v have continuous partial derivatives of all orders at that point.

(iii) **(Cauchy's Inequality)** Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R. If M_R denotes the maximum value of $|f(z)|$ on C_R then $|f^n(z_0)| \leq \frac{n! M_R}{R^n}, n = 1,2,3, \dots$

Unit-III

1. If the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for $z = z_1(\neq z_0)$, then it is absolutely convergent for each $z \in B(z_0, R_1)$ where $R_1 = |z_1 - z_0|$
2. If z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ then show that the series must be uniformly convergent in the closed disk $|z - z_0| \leq R_1$, where $R_1 = |z_1 - z_0|$.
3. A power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ represents a continuous function $S(z)$ at each point inside its circle of convergence $|z - z_0| = r$
4. Let C be a simple closed curve in the interior of the disc of convergence of the power series $S(z) = \sum_{n=0}^{\infty} a_n\,(z-z_0)^n$ and let $g(z)$ be any function which is continuous on C. Then the series $\sum_{n=0}^{\infty} g(z)a_n(z-z_0)^n$ can be integrated term by term over C and

$$\int_C g(z)S(z)dz = \sum_{n=0}^{\infty} \int_{n=0}^{\infty} g(z)a_n(z-z_0)^n dz.$$

5. Let C be a simple closed curve in the interior of the disc of convergence of the power series $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, S'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1.}$(Term by term differentiation of Power Series in the interior of its disk of convergence)
6. If a series $\sum a_n\,(z-z_0)^n$ converges to $f(z)$ at all ponts within the disc of convergence $|z - z_0| < R$ then it is the Taylor series expansion for f centered at z_0. (Uniqueness of Taylor series expansion)
7. State Laurent’s Theorem. (with problems on Laurent’s expansion in different domains)
8. State Cauchy’s Residue Theorem (with problems)
9. Define

(a) An isolated singular point (b) a removable singularity
(c) a pole (d) an essential singularity

with problems

Note :2 results may be combined for appropriate weightage

Problems may get added or theory bits may get combined /shuffled depending on the weightage

Practical no 1. Normal subgroups and Quotient groups

1. Let $H_1 = \{I, (12)\}$ and $H_2 = \{I, (123), (132)\}$. Then
   (a) H_1, H_2 are normal subgroups of S_3.
   (b) H_1 is a normal subgroup of S_3 but H_2 is not a normal subgroup of S_3.
   (c) H_1, H_2 are not normal subgroups of S_3
   (d) H_2 is a normal subgroup of S_3 but H_1 is not a normal subgroup of S_3.

2. Let $H_1 = \{\sigma \in S_n : \sigma(n) = n\}, H_2 = \{\sigma \in S_n : \sigma(k) = k,$ for some $k,\ 1 \leq k \leq n\}$. Then
   (a) H_1, H_2 are normal subgroups of S_n.
   (b) H_1 is a normal subgroup of S_n but H_2 is not a normal subgroup of S_n.
   (c) H_1, H_2 are not normal subgroups of S_n
   (d) H_2 is a normal subgroup of S_n but H_1 is not a normal subgroup of S_n.

3. Let $G = \dfrac{\mathbb{Z}}{20\mathbb{Z}}, H = \dfrac{4\mathbb{Z}}{20\mathbb{Z}}$ (under addition). Then order of quotient group $\dfrac{G}{H}$ is

   (a) 4 (b) ∞ (c) 5 (d) 20

4. Let H be a normal subgroup of G. Let $|aH| = 3$ in $\frac{G}{H}$ and $\circ(H) = 10$, then order of a is
   (a) 1 (b) 30
   (c) one of 3, 6, 15 or 30 (d) none of these.

5. Let G be a group of order 5. If $\Phi : \mathbb{Z}_{30} \to G$ is a group homomorphism, then ker Φ has order
   (a) 5 (b) 30 or 6 (c) 30 or 5 (d) 1

6. Let G be a finite group. If $f_1 : G \to \mathbb{Z}_{10}$ and $f_2 : G \to \mathbb{Z}_{15}$ are onto group homomorphisms, then order of G is
   (a) $30k$, where $k \in \mathbb{N}$ (b) 5^k, where $k \in \mathbb{N}$ (c) 10 or 15 (d) 5

7. In the quotient group $\dfrac{\mathbb{Z}_{18}}{<\bar{6}>}$ (under addition), the order of the element $\bar{5}+ <\bar{6}>$ is

   (a) 5 (b) 6 (c) 2 (d) 3

8. Let H be a subgroup of order 29 of a group G. If K is a subgroup of H, then
   (a) K is abelian and normal subgroup of G.
   (b) K is normal subgroup of H.
   (c) K is cyclic but may not be a normal subgroup of H .
   (d) H is normal subgroup of G and K is normal subgroup G.

9. Let $G = GL_2(\mathbb{R}), K = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}, ad \neq 0\right\}, H = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R}\right\}$. Then

(a) H is a normal subgroup of K and K is a normal subgroup of G.

(b) H is a normal subgroup of K but K is not a normal subgroup of G.

(c) H is a not normal subgroup of K but K is a normal subgroup of G.

(d) None of these.

10. Let H be a normal subgroup of a finite group G. If $|H| = 2$ and G has an element of order 3 then

(a) G has a cyclic subgroup of order 6.

(b) G has a non-abelian subgroup of order 6.

(c) G has subgroup of order 4.

(d) None of these.

11. Let G be a group of order 30. If $Z(G)$ has order 5, then
(a) $\frac{G}{Z(G)}$ is cyclic. (b) $\frac{G}{Z(G)}$ is abelian but not cyclic.
(c) $\frac{G}{Z(G)}$ is non-abelian. (d) None of these.

12. Let $G = GL_2(\mathbb{R})$, $H = \{A \in G : detA \in \mathbb{Q}\}$, then
(a) H is a normal subgroup of G. (b) H is not a subgroup of G.
(c) H is a subgroup which is not normal in G. (d) $H \subseteq Z(G)$.

13. Let $G = GL_2(\mathbb{R})$, $H = \{A \in G : detA = 2^m 3^n, \text{ for some } m, n \in \mathbb{Z}\}$, then
(a) H is a normal subgroup of G. (b) H is not a subgroup of G.
(c) H is a subgroup which is not normal in G. (d) $H \subseteq Z(G)$.

14. Let $G = U(16)$, $H = \{\bar{1}, \bar{15}\}$, $K = \{\bar{1}, \bar{9}\}$, then

(a) H, K are isomorphic groups and $\frac{G}{H}, \frac{G}{K}$ are isomorphic groups.

(b) H, K are not isomorphic groups but $\frac{G}{H}, \frac{G}{K}$ are isomorphic groups.

(c) H is not isomorphic to K.

(d) $\frac{G}{H}, \frac{G}{K}$ are not isomorphic groups.

15. Let $H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in 2\mathbb{Z} \right\}$, $G = M_2(\mathbb{Z})$, under addition of 2×2 matrices.
The quotient group $\frac{G}{H}$ has
(a) 4 elements (b) 16 elements (c) 12 elements (d) 8 elements

16. Let $G = D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}, a^4 = e = b^2, aba = b, H = \{e, b, a^2b, a^2\}, K = \{e, b\}$
(a) K is normal in H and H is normal in G. (b) K is not normal in H.
(c) K is normal in G. (d) H is not normal in G.

17. The quotient group $\left(\dfrac{\mathbb{Q}}{\mathbb{Z}}, +\right)$ is
    (a) an infinite group in which only identity is of finite order.
    (b) is an infinite cyclic group of finite index.
    (c) an infinite group in which every element is of finite order.
    (d) None of these.

Practical 1 Descriptive Question

1. Let $G = GL_2(\mathbb{R}), K = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}, ad \neq 0\right\}$. Prove or disprove: K is normal subgroup of G.

2. Let $G = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}, ad \neq 0\right\}, H = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R}\right\}$. Prove that (i) H is a normal subgroup of G. (ii) $\dfrac{G}{H}$ is abelian.

3. Find the order of $\bar{5} + \langle \bar{14} \rangle$ in $\dfrac{\mathbb{Z}_{42}}{\langle \bar{14} \rangle}$.

4. Find the order of $\bar{14} + \langle \bar{8} \rangle$ in $\dfrac{\mathbb{Z}_{24}}{\langle \bar{8} \rangle}$.

5. In the following examples show that K is a normal subgroup of H and H is a normal subgroup of G, but K is not a normal subgroup of G.
    (i) $G = D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}, a^4 = e = b^2, aba = b, H = \{e, b, a^2b, a^2\}$, $K = \{e, b\}$.
    (ii) $G = A_4, K = \{I, (12)(34), (13)(24), (14)(23)\}, H = \{I, (12)(34)\}$.

6. Let $\mathbb{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}, i^2 = j^2 = k^2 = -1 = ijk$. Show that
    (i) $\mathbb{Z}(\mathbb{Q}_8) = \{1, -1\}$.
    (ii) Every subgroup of $\mathbb{Q}_8$ is normal in $\mathbb{Q}_8$.

7. Let $H = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in 2\mathbb{Z}\right\}, G = M_2(\mathbb{Z})$, under addition of 2×2 matrices. Find order of the quotient group $\dfrac{G}{H}$ and describe $\dfrac{G}{H}$.

8. Let $G = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}, ad \neq 0\right\}, H = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R}\right\}$. Prove that H is a normal subgroup of G and $\dfrac{G}{H} \cong (\mathbb{R}^+, \dot{})$. (the group of positive real numbers under multiplication).

9. Show that $\dfrac{\mathbb{R}^*}{\{1, -1\}} \cong \mathbb{R}^+$, for the multiplicative groups $\mathbb{R}^* = \mathbb{R} - \{0\}$, $\mathbb{R}^+$ of positive reals.

10. Show that A_4 has no subgroup of order 6.

11. Show that order of each element of the quotient group $\dfrac{\mathbb{Q}}{\mathbb{Z}}$ is finite.

12. Let G be a cyclic group of order 36 generated by a. Let $H = \langle a^6 \rangle$. Describe the quotient group $\dfrac{G}{H}$.

13. $G = A_4, K = \{I, (12)(34), (13)(24), (14)(23)\}, H = \{I, (12)(34)\}$. Show that $\dfrac{A_4}{H} \cong A_3$.

14. Let H be a normal subgroup of S_4, $\circ(H) = 4$. Prove that $\dfrac{S_4}{H} \cong S_3$.

15. Show that $\dfrac{\mathbb{Q}}{\mathbb{Z}}$ has a unique subgroup of order n for each positive integer n.

16. Let G be a finite abelian group of order n. If $x^3 = e\ \forall x \in G$, show inductively that the order of G is 3^k for some $k \in \mathbb{N} \cup \{0\}$.

17. Let K be a cyclic subgroup of a group G which is normal in G. Show that any subgroup H of K is a normal subgroup of G.

18. Let G be a subgroup s.t. $(ab)^n = a^n b^n$ for some position integer n. Show that $G(n) = \{x^n \,/\, x \in G\}$ is a normal subgroup of G.

19. Let H and K be subgroup of a group G such that $H \cap K = \{e\}$ then show that $hk = kh$, $h \in H$, $k \in K$.

20. Suppose $G/Z(G)$ is cyclic then prove that G is Abelian. Further if G is a group of order 30 and $Z(G)$ has order 5 Show that $G/Z(G)$ is cyclic.

21. Let H be a normal subgroup of G of order 2. Show that $H \subseteq Z(G)$. Further if G is of order 10 show that G is Abelian.

22. If H is a subgroup of G such that $x^2 \in H$ for each $x \in G$ then show that H is a subgroup of G and G/H is Abelian.

23. Prove that the map $\theta : GL_2(\mathbb{R}) \to (\mathbb{R}^*, \cdot)$ given by $\theta(A) = det\, A$ is an onto homomorphism. Prove $SL_2(\mathbb{R})$ is a normal subgroup of $GL_2(\mathbb{R})$.

24. Let G be a subgroup and $H = \{g^2 \,/\, g \in G\}$ is a subgroup of G. Show H is normal in G.

25. Let H be a normal subgroup of a finite group G. If G/H has an elements of order n show that G has an element of order n.

26. Let $G =< a >$ be a cyclic group of order 21. Let$H =< a^7 >$. Find the order of element a^5H in the quotient group G/H.

27. Let $G =< a >$ be a cyclic group of order 24. Let $H =< a^{12} >$ and $K =< a^6 >$.
    (i) In G/H, find orders of a^2H, a^3H, a^4H, a^5H.
    (ii) In G/K, find orders of a^2K, a^3K, a^4K, a^5K.

28. Show that the map $\phi : \mathbb{Q} \to S_1$ defined by $\phi(m/n) = e^{2\pi mi/n}$, where $m/n \in \mathbb{Q}, (m,n) = 1$ and $S^1 = \{z \in \mathbb{C} \ |z|^2 < 1\}$ is a homomorphism of groups $(\mathbb{Q}, +)$ and $(S^1, \cdot)$. Find ker ϕ ,Im ϕ.

Practical no 2. Cayley's theorem and external direct product of groups

1. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has
(a) 3 subgroups of order 2. (b) 7 subgroup of order 2 (c) 6 subgroups of order 2. (d) 9 subgroups of order 2.

2. The order of any non-identity element in $\mathbb{Z}_3 \times \mathbb{Z}_3$ is
(a) 3 (b) 9 (c) 6 (d) none of these.

3. Which of the following statements is false?
(a) $\mathbb{Z}_3 \times \mathbb{Z}_5$ is isomorphic to $\mathbb{Z}_{15}$ (b) $\mathbb{Z}_3 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$
(c) $\mathbb{Z}_9 \times \mathbb{Z}_9$ is isomorphic to $\mathbb{Z}_{27}$ (d) $\mathbb{Z}_4 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_{12}$

4. The group $S_3 \times \mathbb{Z}_2$ is isomorphic to
(a) $\mathbb{Z}_{12}$ (b) A_4 (c) D_6 (d) $\mathbb{Z}_6 \times \mathbb{Z}_2$

5. Let $G_1 = \mathbb{Z}_4 \times \mathbb{Z}_{15}$ and $G_2 = \mathbb{Z}_6 \times \mathbb{Z}_{10}$, then
   (a) G_1 and G_2 are cyclic groups of order 60.
   (b) G_1 and G_2 are not cyclic groups.
   (c) G_1 is cyclic but G_2 is not cyclic group.
   (d) G_1 is not cyclic but G_2 is a cyclic group.

6. . Which is true about groups?
   (a) $\mathbb{Z}_4 \times \mathbb{Z}_2$ is isomorphic to $V_4 \times \mathbb{Z}_2$.
   (b) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to $V_4 \times \mathbb{Z}_2$.
   (c) $V_4 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$.
   (d) D_4 (the dihedral group of order 8) is isomorphic to Quaternion group Q_8 of order 8.

7. A group of order n is isomorphic to
(a) a subgroup of $\mathbb{Z}_n \times \mathbb{Z}_n$. (b) a subgroup of A_n.
(c) a subgroup of D_n. (d) a subgroup of $\mathbb{Z}_{2n}$

8. $\mathbb{Z}_3$ is isomorphic to the following subgroup of S_3
(a) $<(12)>$. (b) $<(13)>$ (c) A_3 (d) S_3 itself.

9. A group of order 4 in which every element satisfies the equation $x^2 = e$ is isomorphic to
(a) $\mathbb{Z}_2 \times \mathbb{Z}_2$. (b) μ_4, the group of forth roots of unity under multiplication.
(c) $(\mathbb{Z}_4, +)$ (d) $\{\bar{1}, \bar{3}, \bar{7}, \bar{9}\}$.

10. The smallest positive integer n for which there are two non-isomorphic groups of order n equals.
(a) 2 (b) 4 (c) 6 (d) 8

11. For each positive integer n,
(a) There is a cyclic group of order n. (b) There are two non-isomorphic groups of order n.
(c) There is a non-abelian group of order n. (d) The number of non-isomorphic groups of order n is equal to n

12. A non-cyclic group of order 6 is isomorphic to
(a) $\mathbb{Z}_3 \times \mathbb{Z}_2$ (b) μ_6, the group of sixth roots of unity under multiplication.
(c) $U(14) = \{\bar{1}, \bar{3}, \bar{5}, \bar{9}, \bar{11}, \bar{13}\}$. (d) S_3

13. Let $G_1 = \mathbb{Z}_3 \times \mathbb{Z}_5, G_2 = \mathbb{Z}_3 \times \mathbb{Z}_9$. Then
    (a) G_1 is isomorphic to $\mathbb{Z}_{15}$ and G_2 is isomorphic to $\mathbb{Z}_{27}$.
    (b) G_1 and G_2 are not isomorphic to $\mathbb{Z}_{15}$, $\mathbb{Z}_{27}$ respectively.
    (c) G_1 is not isomorphic to $\mathbb{Z}_{15}$ but G_2 is isomorphic to $\mathbb{Z}_{27}$
    (d) G_1 is isomorphic to $\mathbb{Z}_{15}$ but G_2 is not isomorphic to $\mathbb{Z}_{27}$

14. The number of elements of order 4 in $\mathbb{Z}_8 \times \mathbb{Z}_4$ is
(a) 4 (b) 8 (c) 20 (d) 16

15. Consider the following groups i) $\mathbb{Z}_4$ ii) $U(10)$ ii) $U(8)$ iv) $U(5)$. The only non-isomorphic group among them is
(a) $U(8)$ (b) $U(10)$ (c) $\mathbb{Z}_4$ (d) All are isomorphic.

16. Consider the following groups i) S_3 ii) μ_6 ii) $\mathbb{Z}_6$ iv) $\mathbb{Z}_2 \times \mathbb{Z}_3$ v) $U(9)$. The only non-isomorphic group among them is
(a) S_3 (b) μ_6 (c) $\mathbb{Z}_2 \times \mathbb{Z}_3$ (d) $S_3 \simeq U(9)$ and $\mu_6, \mathbb{Z}_6, \mathbb{Z}_2 \times \mathbb{Z}_3$ are isomorphic. .

17. If for positive integers m, n have $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to$(\mathbb{Z}_{mn}, +)$ then which is not true,
    (a) m, n are relatively prime.
    (b) m, n are odd.
    (c) m, n are prime.
    (d) $m = p^r, n = q^s$ for primes p, q and $r, s \in \mathbb{N}$.

18. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ and $H = \mathbb{Z}_4 \times \{\bar{0}, \bar{1}\}, K =< (\bar{1}, \bar{2}) >$ be subgroups of G Then
(a) G/H is isomorphic to G/K (b) G/H is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$
(c) H and K are isomorphic. (d) none of these.

19. From the given list of pairs group, pick the pair of non-isomorphic groups
(a) $3\mathbb{Z}/12\mathbb{Z}$ and $\mathbb{Z}_4$ (b) $8\mathbb{Z}/48\mathbb{Z}$ and $\mathbb{Z}_6$
(c) $\mathbb{Z}_4$ and V_4 (d) $(\mathbb{Z} \times \mathbb{Z})/(2\mathbb{Z} \times 2\mathbb{Z})$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$

20. From the given list of pairs of groups, pick the pairs of isomorphic groups
(a) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$ (b) $\mathbb{Z}_8$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$
(c) D_4 and $\mathbb{Z}_4 \times \mathbb{Z}_2$ (d) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $V_4 \times \mathbb{Z}_2$

Practical 2 Descriptive Question

1. (a) Find all subgroup of order 2 in the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
   (b) Find all subgroups of order 4 in the group $\mathbb{Z}_4 \times \mathbb{Z}_4$.
   (c) Prove or disprove: $\mathbb{Z} \times \mathbb{Z}$ is a cyclic group.

2. (a) Find a subgroup of S_4 isomorphic to i) $\mathbb{Z}_4$ ii) V_4.
   (b) Find a subgroup of S_6 isomorphic to Z_6.

3. Find the left Cayley representation of S_3 in S_6.

4. Find the Cayley representation of $\mathbb{Z}_3$ in S_3.

5. Check whether ,
   (a) $\mathbb{Z}_3 \times \mathbb{Z}_9$ and $\mathbb{Z}_{27}$ are isomorphic groups.
   (b) $\mathbb{Z}_3 \times \mathbb{Z}_5$ and $\mathbb{Z}_{15}$ are isomorphic groups.

6. Show that $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(a, b) = a - b$ is a group homomorphism. Find Ker ϕ and describe the set $\phi^{-1}(3)$.

7. Let $G_1 \times G_2$, where $G_1 = (\mathbb{Z}_4, +), G_2 = \{\bar{1}, \bar{3}\}$ modulo 4 under multiplication. Let $H =< (\bar{2}, \bar{3}) > K =< (\bar{2}, \bar{1}) >$ be subgroups of G. List elements in H and $K, G/H$ and G/K. Show that H is isomorphic to K but G/H is not isomorphic to G/K.

8. Show that $\mathbb{Z}_8 \times \mathbb{Z}_4$ and $\mathbb{Z}_{8,00,000} \times \mathbb{Z}_{4,00,000}$ have same number of elements of order 4.

9. Find all subgroups of order 4 in $\mathbb{Z}_4 \times \mathbb{Z}_4$.

10. Find the number of elements of order 2 in $\mathbb{Z}_{2,00,000} \times \mathbb{Z}_{4,00,000}$.

11. Find a subgroup of $\mathbb{Z}_{12} \times \mathbb{Z}_4 \times \mathbb{Z}_{15}$ of order 9.

12. Let m, n be fixed positive integers. Consider the map $\phi_{m,n} : \mathbb{Z} \to \mathbb{Z}_m \times \mathbb{Z}_n$ defined by $\phi_{m,n}(x) = (x\ mod\ m, x\ mod\ n)$. Show that $\phi_{m,n}$ is a group homomorphism. Find ker $\phi_{m,n}$.

NES RATNAM COLLEGE OF ARTS, SCIENCE & COMMERCE, BHANDUP-78

(i)

Paper II **Practical No.3** **Semester VI**

Rings, Subrings, Integral Domains

Objective Questions

1. Let R be a ring and a, b be non-zero elements of R. The equation $ax = b$ has
   a) a unique solution in R
   b) at most one solution in R
   c) may have more than one solution in R
   d) None of the above

2. The group of units of the ring $\mathbb{Z}_{25}$ is
   a) $\{\bar{1}, 3, \bar{5}, \bar{7}, \overline{23}\} mod\ 25$
   b) $\{\bar{1}\bar{2}, \bar{3}, \bar{4}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \overline{11}, \overline{12}, \overline{13}, \overline{14}, \overline{16}, \overline{17}, \overline{18}, \overline{19}, \overline{21}, \overline{22}, \overline{23}, \overline{24}\}\ mod\ 25$
   c) $\{\bar{1}, \bar{4}, \bar{8}, \overline{12}, \overline{16}, \overline{20}\} mod\ 25$
   d) $\{\bar{1}, \bar{3}, \bar{6}, \bar{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}\}\ mod\ 25$
3. The group of units of a ring is
   a) abelian but may not be cyclic (b) Cyclic (c) may not be abelian (d) finite
4. Consider the ring $M_2(\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a, b, c, d \in \mathbb{Z}\right\}$ under addition and multiplication of 2×2 matrices.
   Then $A \in M_2(\mathbb{Z})$ is a unit iff
   (a) $\det A \neq 0$ (b) $\det A = 1$ (c) $\det A > 0$ (d) $\det A = \pm 1$
5. Consider the following rings
   (i) $(\mathbb{Z}_5, +, \cdot)$ (ii) $(\mathbb{Z}_{15}, +, \cdot)$
   (iii) $\mathbb{Z} \times \mathbb{Z}$ under component wise addition and multiplication (iv) $\mathbb{R}[x]$
   Then
   (a) (i), (iv) have no proper zero divisors. (b) (i), (iii) have no proper zero divisors
   (c) (i), (ii) have no proper zero divisors (d) (i), (iii), (iv) have no proper zero divisors
6. The number of units in the ring $\mathbb{Z}_{20}$ is
   (a) 5 (b) 6 (c) 7 (d) 8
7. Which of the following is a subring of $\mathbb{Q}(+, \cdot)$
   (i) $R = \left\{\frac{a}{b}\ ; a, b \in \mathbb{Z}\ (a, b) = 1,\ b \text{ is not divisible by } 3\right\}$
   (ii) $R = \left\{\frac{a}{b}\ ; a, b \in \mathbb{Z}\ (a, b) = 1,\ b \neq 0, b \text{ is divisible by } 3\right\}$
   (iii) $R = \{x^2 : x \in \mathbb{Q}\}$
   (iv) $R = \left\{\frac{a}{b} = a, b \in \mathbb{Z}, b \neq 0\ \ (a, b) = 1\ \ a \text{ is divisible by } 3\right\}$
   (a) (i) and (iv) (b) (ii) and (iv) (c) (i) and (ii) (d) only (i)
8. Let R and S be rings. Consider $R \times S$ under componentwise addition and multiplication
   (i) If R and S are integral domains then $R \times S$ is an integral domain.
   (ii) $R \times S$ is an integral domain iff R and S are integral domains.
   (iii) $R \times S$ is not an integral domain whatever R, S may be.
   (iv) $R \times S$ is not commutative even if R, S are commutative.

9. Let R be an integral domain, then the equation $x^2 = 1$ has
   (a) exactly two solutions (b) may not have any solution
   (c) may have more than two solutions (d) None of these
10. Consider the following rings
   (i) $\mathbb{Z}_{18}$ (ii) $\mathbb{Z}_{12}$ (iii) $\mathbb{Z}_{10}$ (iv) $\mathbb{Z}_{14}$ Then
   (a) (i), (ii), (iii) ,(iv) have nilpotent elements (b) (i), (ii) have nilpotent elements
   (c) (iii), (iv) have nilpotent elements (d) None of these
11. In an integral domain the number of elements which are their own inverses is
   (a) 1 (b) 1 or 2 (c) infinitely way (d) cannot say

12. In a ring $(\mathbb{Z}_n, +, \cdot)$ where n is a positive integer > 1
   (i) $\bar{a}^2 = \bar{a} \Longrightarrow \bar{a} = 0$ or $\bar{a} = \bar{1}$ for $\bar{a} \in \mathbb{Z}_n$.
   (ii) $\bar{a} \cdot \bar{b} = \bar{0} \Longrightarrow \bar{a} = \bar{0}$ or $\bar{b} = 0$ for $\bar{a}, \bar{b} \in \mathbb{Z}_n$.
   (iii) $\bar{a} \cdot \bar{b} = \bar{a} \cdot \bar{c}, \bar{a} \neq = 0 \Longrightarrow \bar{b} = \bar{c}$ for $\bar{b}, \bar{c} \in \mathbb{Z}_n$. Then,
   (a) the statements (i), (ii), (iii) are true.
   (b) the statements (i) is true but (ii), (iii) may not be true.
   (c) the statements (i), (ii), (iii) are true if n is prime.
   (d) None of the above

13. If R is a ring and a, b are zero divisors in R, then
   (a) $a + b$ is always a zero divisor (b) $a + b$ is not a unit in R
   (c) $a + b$ may not be a zero divisor (d) None of these

14. In the ring $R = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z}_2\right\}$, the number of non-zero divisors is
   (a) 6 (b) 7 (c) 3 (d) None of these

15. If x is an idempotent element in $\mathbb{Z}_n (x^2 = x)$, then
   (a) $1 - x$ is a unit (b) $1 + x$ is a unit (c) $1 - x$ is a idempotent (d) None of these

16. Let R be a commutative ring such that $a^2 = 0 \Longrightarrow a = 0 \; \forall \; a \in R$. then
   (a) R has no proper zero divisors (b) R has no nilpotent elements
   (c) R is an integral domain but not a field (d) None of these

17. Consider the rings $R_1 = (\mathbb{Z}_{10}, +, \cdot), R_2 = (\mathbb{Z}_{23}, +, \cdot), R_3 = M_2(\mathbb{Z}), R_4 = \mathbb{Z} \times \mathbb{Z}$ under component wise addition and multiplication.
   (a) R_1, R_2, R_3, R_4 are all integral domains (b) Only R_2, R_3, R_4 are integral domains
   (c) R_2 is an integral domain (d) R_2, R_4 are integral doamins

18. Let R be an integral domain of characteristic p. Then,
   (a) $(x + y)^m = x^m + y^m \;\; \forall x, y \in R$ if and only if $m = p$.
   (b) $(x + y)^m = x^m + y^m \;\; \forall x, y \in R$ if and $m = kp$.
   (c) $(x + y)^{p^n} = x^{p^n} + y^{p^n} \;\; \forall x, y \in R$ and for all $n \in \mathbb{N}$.
   (d) None of the above
19. Consider the subset $S = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}\}$ of $\mathbb{Z}_{10}$.
   (a) S is a subring of $\mathbb{Z}_{10}$

(b) S is not a subrings of $\mathbb{Z}_{10}$.
(c) S is a subrings with multiplicative identity $\overline{6}$.
(d) S is a ring with multiplicative identity $\overline{6}$.

20. Let R be a commutative ring such that $a^2 = 0 \Longrightarrow a = 0$ for $a \in R$. then,
(a) R has no proper zero divisors (b) R has no nilpotent elements
(c) R is a an integral domain but not a field (d) None of these

21. Let R be a ring in which $x^2 = x$ for all $x \in R$. Then,
(a) R is an integral domain with characteristic 3.
(b) R is field with characteristic 3.
(c) Characteristic of R is 2.
(d) None of these

22. In a ring $R = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z}_2\right\}$, the number of zero divisors are
(a) 6 (b) 7 (c) 3 (d) None of these

23. The characteristics of the ring $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$ under component wise addition and multiplication is
a) 180 b) 3 c) 60 d) 5.

DESCRIPTIVE QUESTIONS

1) Let $R(+,\cdot)$ be a ring. Show that $R(\oplus,\odot)$ is a ring where
$$a \oplus b = a + b - 1_R, \qquad a \odot b = a + b - ab$$
2) a) Let R be a ring. If $x^3 = x \ \forall\, x \in R$, show that R is commutative .
b)Let R be a ring. If $x^4 = x \ \forall\, x \in R$, show that R is commutative.
c)Let R be a ring in which $ab = ca \Rightarrow b = c$ for $a, b, c \in R,\ a \neq 0$. Show that R is commutative.
d) If R is ring with more than one element. If $ax = b$ has a solution for all non-zero $a \in R$ and for all $b \in R$, then show that R is a division ring.

3) Show that $\mathbb{Z} \times \mathbb{Z}$ under componentwise addition and multiplication is a ring. Is it an integral domain? Justify your answer.
4) Show that $R_p = \{m/n : m, n \in Z\,; (m, n) = 1; p \nmid n\}$ for a fixed prime p is a ring.
5) Show that $\mathbb{Z}[i] + \{a + bi : a, b \in \mathbb{Z}\}$ is a integral domain.
6) a) Show that a ring that is cyclic under addition is commutative.
b)Let R be a ring having 6 elements. Show that R is commutative. Is R an integral domain? Justify your answer
7) Show that every non-zero element in $\mathbb{Z}_n$ is either a unit or a zero divisor.
8) Let R be an integral domain and $a, b \in R$
(i) If $a^7 = b^7, a^{12} = b^{12}$ show that $a = b$
(ii) If $a^m = b^m,\ a^n = b^n,\ m, n \in \mathbb{N}\ (m, n) = 1$, then $a = b$
9) Let $H = \left\{\begin{pmatrix} z & w \\ -w & \bar{z} \end{pmatrix}, z, w \in C\right\}$ Show that H is a non-commutative subring of $M_2(C)$ which is a division ring.

10) Show that $R = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z}_7 \right\}$ under usual matrix addition and multiplication and modulo 7 addition and multiplication of entries is commutative ring. Is R an integral domain? Justify your answer.

What happens If $R = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{Z}_5 \right\}$.

11) a)Let R be a commutative ring. If u is a unit and a is nilpotent in R. Show that $u + a$ is unit.

b)If a, b are nilpotent elements of a commutative ring, show that $a + b$ is also nilpotent. Give an example to show that this may fail if the ring is not commutative.

c)Let x be a non-zero element of a ring R. If there exists a unique $y \in R$ such that $xyx = x$, then show that x is invertible in R.

d)Determine all zero divisors, units and idempotent and nilpotent elements of the following rings-

(i) $(\mathbb{Z}_{18}, +, \cdot)$ (ii) $\mathbb{Z}_3 \times \mathbb{Z}_6$ under component wise addition and multiplication.

(iii) $F \times F$ where F is a field (iv) $(P(X), +, \cap)$

e) Find zero divisors, idempotent, nilpotent elements in $\mathbb{Z}_3 \oplus \mathbb{Z}_6$.

f) Find two elements a and b in a ring such that both a and b are zero-divisors, $a + b \neq 0$, and $a + b$ is not a zero-divisor.

12) a) Let a belong to a ring R with unity and $a^n = 0$ for some positive integer n. (Such an element is called $nilpotent$.) Prove that $1 - a$ has a multiplicative inverse in R. (Hint: Consider $(1 - a)(1 + a + a^2 + \cdots + a^{n-1})$.]

b)Show that the nilpotent elements of a commutative ring form a subring.

c) Show that 0 is the only nilpotent element in an integral domain.

d) A ring element a is called an idempotent if $a^2 = a$. Prove that the only idempotents in an integral domain are 0 and 1.

e) Find a zero-divisor and a nonzero idempotent other than 1 in $Z_5[i] = \{a + bi \,|a, b \in Z_5, i^2 = -1\}$.

f) If a is an idempotent in Z_n, show that $1 - a$ is also an idempotent.

14. Let R be an integral domain with characteristic 2. Show that-

(a) $(a + b)^2 = a^2 + b^2 \; \forall a, b \in R$.

(b) $S = \{a \in R \; a^2 = a\}$ is a subrings of R.

15. Determine all subrings of the following rings

(a) $(\mathbb{Z}_{12}, +, \cdot)$ (b) $(\mathbb{Z}_7, +, \cdot)$ (c) $(\mathbb{Z}, +, \cdot)$

16. In the following examples, show that S is a subring of given ring R

(i) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \; s.t. \; a + c = b + d \right\}, R = M_2(\mathbb{R})$.

(ii) $S = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\} \; R = M_2(\mathbb{R})$

(iii) $S = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \in \mathbb{Q} \right\} \; R = M_2(\mathbb{Q})$

(iv) $S = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, (a, b) = 1,\ b\ odd\right\} R = \mathbb{Q}$

(v) $S = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\ (a, b) = 1, b\ even\right\}\ R = \mathbb{Q}$

17. Show that $\mathbb{Z}[\sqrt{2}]$ has infinitely many units. every $\left(3 + 2\sqrt{2}\right)^n$is a unit where n is a positive integer.
18. a) Prove or disprove: If R is a ring with characteristic P, R is finite.
    b)Consider the ring $R = \{0,2,4,6,8,10\}$ under addition and multiplication modulo 12. What is the characteristic of R.
    c)Let R be a ring in which $x^4 = x\ \forall \in R$Find characteristic of R.
19. Give example –
    (a) of a finite ring which is non-commutative
    (b) of a ring R such that $a^2 = a$ for all $a \in R$.
    (c) of a commutative ring without zero-divisors that is not an integral domain.
20. Let x and y belong to an integral domain of prime characteristic p.
    (a) Show that $(x + y)^p = x^p + y^p$.
    (b) Show that, for all positive integers n, $(x + y)^{p^n} = x^{p^n} + y^{p^n}$.
    (c) Find elements x and y in a ring of characteristic 4 such that $(x + y)^4 \neq x^4 + y^4$.

(d) Let R be an ID of characteristic 2 Show $(x + y)^2 = x^2 + y^2$. further show $(x + y)^{2^n} = x^{2^n} + y^{2^n}\ \forall n \in \mathbb{N}$.

NES RATNAM COLLEGE OF ARTS, SCIENCE & COMMERCE, BHANDUP-78

Paper II **Practical No.4** **Semester VI**

Ideals, Quotient Rings, Homomorphism and Isomorphisms of rings

Objective Questions

1. Consider the ring $\mathbb{Z} \times \mathbb{Z}$ under component wise addition and multiplication.
   Let $I = \{(a, -a) = a \in \mathbb{Z}\}$
   $J = \{(a, 0) = a \in \mathbb{Z}\}$
   (a) I and J are ideal of $\mathbb{Z} \times \mathbb{Z}$ (b) I and J are subrings of $\mathbb{Z} \times \mathbb{Z}$
   (c) neither I nor J are ideal of $\mathbb{Z} \times \mathbb{Z}$ (d) J is a subring of $\mathbb{Z} \times \mathbb{Z}$, but I is not
2. Consider the ring $M_2(\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}\right\}$
   Let $I = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are divisible by } 5\right\}$. Then
   (a) I is a subring of $M_2(\mathbb{Z})$ but not an ideal of $M_2(\mathbb{Z})$
   (b) I is an ideal of $M_2(\mathbb{Z})$ but not a subring.
   (c) I is not an ideal of $M_2(\mathbb{Z})$
   (d) I is both a subring and an ideal of $M_2(\mathbb{Z})$.
3. Consider the ideals $I = 10\,\mathbb{Z}$ and $J = 12\mathbb{Z}$ then
   (a) $I + J = 22\mathbb{Z}$ $\ IJ = 120\mathbb{Z}$ (b) $I + J = 2\mathbb{Z}$ $\ IJ = 60\mathbb{Z}$
   (c) $I + J = 2\mathbb{Z}$ $\ IJ = 120\mathbb{Z}$ (d) None of these
4. In the ring of integers Z, consider the ideals $I = 4Z + 6Z, J = mZ + nZ,\ m, n \in IN$. Then,
   a) $I = 2Z, \quad J = dZ, \quad where\ d = gcd\ of\ m\ and\ n.$
   b) $I = 24Z, \quad J = mn\,Z$
   c) $I = 12Z, \quad J = \ell\, Z\ where\ \ell = lcm\ of\ m\ and\ n.$
   d) None of these.

5. In the ring of integers Z, consider the ideal $I = (6Z)\,(4Z)$
   a) $I = 24Z$ b) $I = 12Z$ c) $I = 2Z$ d) None of these

6. The number of ring homomorphisms from $Z \oplus \mathbb{Z} \times \mathbb{Z}$ are
   (a) 0 (b) 1 (c) 2 (d) 3
7. The number of ring homomorphisms from Q to itself are
   (a) 1 (b) 2 (c) infinitely many (d) none of these
8. The number of ring homomorphisms from C to itself are
   (a) 1 (b) 2 (c) infinitely many (d) none of these
9. From the following pairs of rings, the isomorphic pair is
   (a) $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{5}]$ (b) $\mathbb{Z}_6 \oplus \mathbb{Z}_4\ and\ \mathbb{Z}_{24}$
   (c) R and C (d) $R = \left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in R\right\}\ and\ C$
10. The Kernel of the ring homomorphisms
   $\phi: \mathbb{R}[x] \to \in$ defined by $\phi\big(f(x)\big) = f(2 + i)$ is
   (a) The principal ideal $(x^2 + 2x + 1)$

(b) the principal ideal $(x-2)$
(c) the principal ideal (x^2-4x+5)
(d) the principal ideal (x^2+4x+1)

11. Consider the following maps from $M_2(\mathbb{Z}_p) \to \mathbb{Z}_p$ defined by
$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a,\ \ g\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a+d,\ \ h\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det A$
(a) f, g, h are all ring homomorphisms.
(b) f is a ring homomorphisms, g and h are not
(c) only h is a ring homomorphism
(d) none of these

12. Consider the following pairs the rings.
(i) $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{5}]$.(ii) $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\sqrt{-5}]$.
(ii) $\mathbb{Q}$ and $\mathbb{R}$.
(iii) (iv) M and $\mathbb{C}$ where $M = \left\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle| a, b \in \mathbb{R}\right\}$.

Then
(a) (i) and (iv) are isomorphic pairs of rings.
(b) (i) and (ii) are isomorphic pairs of rings.
(c) Only (iv) is an isomorphic pair of rings.
(d) (i), (ii) and (iv) are isomorphic pairs of rings.

13. Let $\mathbb{Z}[x]$ and $\mathbb{Z}_n[x]$ denote polynomial rings. The map $\emptyset_n : \mathbb{Z}[x] \to \mathbb{Z}_n[x]$ defined by $\emptyset_n : \mathbb{Z}[x] \to \mathbb{Z}_n[x]$ defined by $\emptyset_n(a_0 + a_1x + \cdots + a_kx^k) = \overline{a_0} + \bar{a}_1x + \ldots. + \bar{a}_kx^k$, where $\bar{a}_i = a_i \bmod n$, for $0 \le i \le k$, is a ring homomorphism only if
(a) n is prime. (b) n is a positive integer (c) n is odd. (d) n is even.

14. The quotient ring $\frac{\mathbb{Z}[i]}{(1+i)}$ is
(a) An integral domain which is not a field. (b) a field having 2 elements.
(c) a field having 5 elements. (d) a ring with proper zero divisors.

15. The kernel of the ring homomorphism $\emptyset : \mathbb{R}[x] \to \mathbb{C}$ defined by $\emptyset(f(x)) = f(2+i)$ is
(a) The principal ideal $(x-2)$.
(b) The principal ideal (x^2-4x+5)
(c) The principal ideal (x^2-4x-5)
(d) The principal ideal (x^2-4x+2).

16. Consider the maps $\pi_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ and $i_1 : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $\pi_1(m, n) = m$, $i_1(m) = (m, 0)$where $\mathbb{Z} \times \mathbb{Z}$ denotes the ring with componentwise addition and multiplication.
(a) π_1 and i_1 are ring homomorphisms
(b) Both π_1 and i_1 are not ring homomorphism
(c) π_1 is a ring homomorphism but i_1 is not a ring homomorphism
(d) i_1 is a ring homomorphism and π_1 is not a ring homomorphism

17. The number of ring homoomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$ are
(a) One (b) ∞ (c) two (d) None of these

.

18. Consider the ring homomorphism $\emptyset : \mathbb{R}[x] \to \mathbb{R}$ defined by

$$\emptyset(a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1 + \cdots + a_n.$$

Then ker $\emptyset$ is

(a) principal ideal (x)

(b) principal ideal $(x+1)$

(c) principal ideal $(x-1)$

(d) None of the above.

DESCRIPTIVE QUESTIONS

1. Check whether following sets are ideals of the ring $\mathbb{Z} \times \mathbb{Z}$ under component wise addition and multiplication.
   (a) $I = \{(a, a) : a \in \mathbb{Z}\}$ (b) $I = \{(2a, 2b) : a, b \in \mathbb{Z}\}$
   (c) $I = \{(2a, 0) : a \in \mathbb{Z}\}$ (d) $I = \{(a, -a) : a \in \mathbb{Z}\}$
2. Check whether of the following are ideals of the polynomial ring $\mathbb{Z}[x]$.
   (a) $I = \{f(x) = a_0 + a_1x + \cdots a_nx^n \in \mathbb{Z}[x]: 3|a_0\}$.
   (b) $I = \{f(x) = a_0 + a_1x + \cdots a_nx^n \in \mathbb{Z}[x]: 3|a_2\}$.
   (c) $I = \{f(x) = a_0 + a_1x + \cdots a_nx^n \in \mathbb{Z}[x]: f(0) = 0\}$.
   (d) $I = \{f(x) = a_0 + a_1x + \cdots a_nx^n \in \mathbb{Z}[x]: \sum_{i=0}^{n} a_i = 0\}$.

3. (a) Let R be a commutative ring and $a \in R$ be non-zero. Show that, annihilator of $a, ann(a) = \{r \in R : ra = 0\}$ is an ideal of R.
   (b) If A, B are ideals of a commutative ring R such that $R = A + B$, show that $A \cap B = AB$.
   (c) Let A and B be ideals of a ring R. If $A \cap B = \{0\}$then show $ab = 0$ when $a \in A$ and $b \in B$.

4. Let $S = \{a + bi \mid a, b \in Z, b \; is \; even\}$. Show that S is a subring of $Z[i]$, but not an ideal of $Z[i]$

5. Show that $I = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \; a, b, c, d \; are \; even \; integers\right\}$ is an ideal of $M_2(\mathbb{Z})$
6. Show that $I = \left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R}\right\}$ is an ideal of the ring $R = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}\right\}$.
7. Show that $I = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \; a, b, c, d \in \mathbb{Z} \; are \; divisible \; by \; 5\right\}$ is an ideal an $M_2(\mathbb{Z})$.
8. Is $I = \{4a + bi : a, b \in \mathbb{Z}\}$ an ideal of $\mathbb{Z}[i]$? Justify your answer.

9. Let R be a ring and I be an ideal of R. Show that $I^m = \left\{\sum_{1=1}^{R} a_{i1}a_{i2} \; a_{in} = a_{ij} \in I, R \in \mathbb{N}\right\}$ is an ideal of R.

10. Let R be a commutative ring and S be the set of all nilpotents elements of R. Show that S forms an ideal of R.Is S a subring of R? Justify your answer.

11. Find the characteristic of $\frac{Z[i]}{\langle 2+i \rangle}$

12. Show that the following are isomorphic:

(a) Rings $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$, $H = \left\{\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} | a, b \in \mathbb{Z}\right\}$.

(b) Rings $R = \left\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} | a, b \in \mathbb{R}\right\}$ and $\mathbb{C}$.

13. Prove or disprove:

(a) The map $\emptyset : M_2(\mathbb{Z}) \to \mathbb{Z}$ defined by $\emptyset\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a$ is a ring homomorphism.

(b) Let $R = \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b, c \in \mathbb{Z}\right\}$. The map $\emptyset : R \to \mathbb{Z}$ defined by $\emptyset\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = a$ is a ring homomorphism.

14. Consider the map $\emptyset : \mathbb{R}[x] \to M_2(\mathbb{R})$ defined by $\emptyset(a_0 + a_1 x + \cdots + a_n x^n) = \begin{bmatrix} a_0 & a_1 \\ 0 & a_0 \end{bmatrix}$. Show that $\emptyset$ is a ring homomorphism. Determine Ker $\emptyset$.

15. Let R be a commutative ring of characteristic p. Show that the map $f : R \to R$ defined by $f(x) = x^p$ is a ring homomorphism.

16. Show that the following maps are ring isomorphism

(a) Let $R = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R}\right\}$ $f: R \to \mathbb{C}$ defined by $f\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a + b\,i$

(b) Let $R = \left\{\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z}\right\}$ $f: R \to \mathbb{C}$ defined by $f\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} = a + b\sqrt{2}$

17. Determine whether the following pairs of rings are isomporphic

(i) $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{5}]$ (ii) $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\sqrt{-5}]$ (iii) $\mathbb{Z}_4 \times \mathbb{Z}_6$ *and* $\mathbb{Z}_{24}$

(iv) $\mathbb{R}(+,\cdot)$ and $\mathbb{Q}(+,\cdot)$ (v) $\mathbb{R}(+,\cdot)$ and $\mathbb{C}(+,\cdot)$

18. Show that the union of ideals of a ring R in a chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq I_{n+1}$ is an ideal.

19. Let R be a commutative ring and I be an ideal in R.

(i) Show that $J = \{x \in R : x\,a = 0 \;\; \forall\, a \in I\}$ is an ideal in R.

(ii) Show that $J = \{x \in R : x^n \in I \text{ for some } n \in \mathbb{N}\}$ is an ideal in R.

20. Find all ideals of the ring $\mathbb{Z}/_{12\mathbb{Z}}$ using correspondence Theorem.

21. Let $R = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, a, b, d \in \mathbb{Z}\right\}$ show that $\phi : R \to \mathbb{Z} \times \mathbb{Z}$ defined by $\phi\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = (a, b)$ is a ring homomorphism. Find ker ϕ.

22. Show that $\mathbb{Z}[i]/(2+i)$ is a finite field, where $(2+i) = \{(2+i)(m + i\,n) : m + in \in \mathbb{Z}[i]\}$

Paper II **Practical No.5** **Semester VI**

Prime and Maximal Ideals, Divisibility in Integral Domains

Objective Questions

1. Let $R = M_2(\mathbb{Z})$ and $I = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are in } \mathbb{Z} \text{ and are divisible by } 5\right\}$ Then
   (a) I is not an ideal. (b) I is a prime ideal but not a maximal ideal.
   (c) I is a maximal ideal (d) I is an ideal but not a prime ideal.
2. Let R be a commutative ring. If 0 is a maximal ideal then
   (a) $R = 0$ (b) R is a finite non-zero ring
   (c) R is a field (d) R is an integral domain which is not a field.
3. The number of maximal ideals in $\mathbb{Z}_{16}$ are
   (a) 4 (b) 2 (c) 1 (d) 3
4. Let $R = M_2(\mathbb{Z}_2)$ and $I = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A : A \in R\right\}$ Then
   (a) I is not an ideal in R. (b) I is a prime ideal which is not maximal.
   (c) I is a maximal ideal (d) I is an ideal but not a prime ideal
5. Let $R = C[0,1], I = \left\{f \in R : f\left(\frac{1}{2}\right) = 0\right\}$ $C[0,1] =$ Ring of continuous real valued functions on $[0,1]$ under pointwise addition and multiplication.
   (a) I is not an ideal in R (b) I is a prime ideal which is not maximal
   (c) I is a maximal ideal (d) I is an ideal but not a prime ideal
6. If R is an integral domain and J is a proper ideal then
   (a) R/I is an integral domain (b) R/I is a field
   (c) R/I is finite (d) R/I may not be commutative
7. Let R be a finite commutative ring. Then
   (a) R is a field (b) 0 is the only proper ideal of R
   (c) every prime ideal is maximal (d) R is an integral domain
8. If P_1 and P_2 are prime ideals in a commutative ring R. then
   (a) $P_1 \cup P_2$ and $P_1 \cap P_2$ are prime ideals in R.
   (b) $P_1 \cap P_2$ may not be a prime ideal in R
   (c) $P_1 \cap P_2$ is a prime ideal in R but $P_1 \cup P_2$ may not be
   (d) None of these

9. Let $S = \{a + bi : a, b \in Z, a, b \text{ are divisible by } 5\}$. Then,
   a) S is not an ideal but is a sub ring of Z[i]
   b) S is an ideal as well as sub ring of Z [i]
   c) S is an ideal of Z [i]
   d) None of these.

10. Consider the ring $M_2(Z) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z\right\}$.
    Let $I = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are even integers}\right\}$
    a) I is not an ideal of $M_2(Z)$.
    b) I is an ideal of $M_2(Z)$ which is not a prime ideal.

b) I is a prime ideal in $M_2(Z)$
d) None of these.

11. Let R be a commutative ring. If $\{0\}$ is a maximal ideal in R, then
 a) R is a finite ring
 b) R is a field
 c) R is an integral domain, which is not a field
 d) None of these

12. The number of maximal ideals in Z_{20} are
 a) 2 b) 4 c) 1 d) None of these

13. Let F be a field. In the ring $F \times F$ under component wise addition and multiplication, the number of maximal ideal are
 a) 1 b) 2 c) 3 d) 0.

14. Let R be a commutative ring. If (0) is the only maximal ideal in R, then
 (a) R is a finite ring (b) R is an integral domain, but not field
 (c) R is a field (d) None of these

15. In the polynomial ring $\mathbb{Z}[x]$, consider $I = \{f(x): f(0) = 0\}$, then
 (a) I is an ideal (b) I is an prime ideal but not maximal ideal
 (c) I is a maximal ideal (d) I is ideal but neither prime ideal nor maximal

16. Let R be a commutative ring, and p_1 and p_2 are prime ideals of R, then
 (a) $P_1 \cup P_2$ and $P_1 \cap P_2$ both and prime ideals of R.
 (b) $P_1 \cap P_2$ is prime ideal of R always but $P_1 \cup P_2$ may not be.
 (c) If $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$ then $P_1 \cap P_2$ is prime ideal of R.
 (d) None of the above

17. Which of the following is irreducible in $\mathbb{Z}[\sqrt{5}]$
 (a) $9 + 4\sqrt{5}$ (b) $1 + \sqrt{5}$ (c) 5 (d) $4 + \sqrt{5}$

18. Which of the following is true in $\mathbb{Z}[\sqrt{-5}]$
 (a) $2 + \sqrt{-5}$ is irreducible but not prime. (b) $2 + \sqrt{-5}$ is prime
 (c) 3 is prime (d) 4 is reducible

19. The number of maximal ideals in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is
 (a) 1 (b) 3 (c) 6 (d) 9

20. Which of the following is prime in $\mathbb{Z}[i]$,
 (a) 2 (b) 5 (c) 17 (d) 3

21. Consider the ring homomorphisms$f_1 : \mathbb{Z}[i] \to \mathbb{Z}_2$ defined by $f(a+bi) = (a-b) mod\ 2$ and $f_2 : \mathbb{Z}[i] \to \mathbb{Z}_5$ defined by $f(a+bi) = (a-2b) mod\ 5$. Then
(a) ker f_1 is a maximal ideal but ker f_2 is not a maximal ideal.
(b) both ker f_1 and ker f_2 are not maximal ideals.
(c) both ker f_1 and ker f_2 are maximal ideals.

22. In the polynomial ring Z[x], consider $I = \{f(x) : f(0) = 0\}$.
a) I is not an ideal in Z [x]
b) I is a prime ideal which is not maximal.
c) I is a maximal ideal
d) I is an ideal which is neither prime nor maximal.

23. Consider the polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n, a_n \neq 0 \ \in \mathbb{R}[x]$, $f(x)$ is a unit in $\mathbb{R}[x]$ iff
(a) each $a_i = 1$ (b) $a_0 \neq 0, a_i = 0$ for $1 \leq i \leq n$
(c) $a_0 = 0, a_n = 1$ (d) $a_n = \pm 1$

24. In the polynomial ring Z[x], consider $I = \{f(x) : f(0) = 0\}$.
a) I is not an ideal in Z [x]
b) I is a prime ideal which is not maximal.
c) I is a maximal ideal
d) I is an ideal which is neither prime nor maximal.

25. The kernel of the ring homomorphism
$\phi : \mathbb{R}[x] \to \mathbb{C}$ defined by $\phi\,(f(x)) = f(1+i)$ is
(a) $(x-1)$ (b) $(x^2 - x + 1)$ (c) $(x^2 - 2x + 2)$ (d) $(x^2 + x + 1)$

26. In the ring $\mathbb{R}[x]$ and $\mathbb{C}[x]$, consider the ideal $I = (x^2 - x + 2)$,
(a) I is a maximal ideal in both $\mathbb{R}[x]$ and $\mathbb{C}[x]$. (b) I is a maximal ideal in $\mathbb{R}[x]$ but not $\mathbb{C}[x]$
(c) I is a maximal ideal in $\mathbb{C}[x]$ but not in $\mathbb{R}[x]$ (d) I is a not maximal ideal in $\mathbb{R}[x]$ and $\mathbb{C}[x]$.

DISCRPTIVE QUESTIONS

1. Let R, S be commutative rings. And $f : R \to S$ be an onto ring homomorphism. Prove that
(i) If P is a prime ideal in S, then $f'(P)$ is a prime ideal in R.
(ii) If M is a maximal ideal in S $f^{-1}(M)$ is a maximal ideal in R.
Do the above results hold if f is not onto? Justify your answer.

2. Prove or disprove:
If R, S are commutative rings and $f : R \to S$ is a ring homomorphism then
(i) P is a prime ideal in $R \Rightarrow f\ (P)$ is a prime ideal in S.
(ii) M is a maximal ideal in $R \Rightarrow f(M)$ is a maximal ideal in S.

3. Prove that
(i) R is an integral domain iff $\{0\}$ is a prime ideal in R.
(ii) R is a field iff $\{0\}$ is a maximal ideal in R.

4. Let R be a ring and I be an ideal of R. Let M be an ideal of R containing I, and let $\bar{M} = M/I$ be the corresponding ideal of R/I. Prove that M is maximal if and only if $\bar{M}$ is maximal.

5. Let $R = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$.Let $\phi : R \to \mathbb{Z}$ be defined by $\phi \begin{pmatrix} a & b \\ b & a \end{pmatrix} = a - b$ Then, Is $\ker \phi$ a prime ideal? Is $\ker \phi$ maximal ideal? Justify your answer.

6. Show that the following ideals are maximal in the indicated ring

(a) $I = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}, a - b \text{ is even}\}$ in $\mathbb{Z}[\sqrt{-5}]$
(b) $\langle x^2 + 1 \rangle$ in $\mathbb{R}[x]$
(c) $I = \{(3x, y): x, y \in \mathbb{Z}\}$ in $\mathbb{Z} \times \mathbb{Z}$
(d) $I = \{f \in R: f(0) = 0\}$ in the ring of continuous function from $\mathbb{R}$ to $\mathbb{R}$
(e) $(\sqrt{2})$ in $\mathbb{Z}[\sqrt{2}]$
(f) I= $\{a + b\, i / a \mod 2 = b \mod 2\}$ in $\mathbb{Z}[i]$

7. Find the maximal ideals of the following rings
a) Z_8 b) Z_{10} c) $IR \times IR$under component wise addition and multiplication.

(d) $\mathbb{Z}_{24}$ (e) $\mathbb{Q}$ (f) $\mathbb{Z} \oplus \mathbb{Z}$

8. Determine the maximal ideals of each of the following
(a) $\mathbb{R} \times \mathbb{R}$ (b) $\mathbb{R}[x]/(x^2)$ (c) $\mathbb{R}[x]/(x^2 - 3x + 2)$ (d) $\mathbb{R}[x]/(x^2 + x + 1)$
(e)$R = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, (a, b) = 1, b \text{ is odd} \right\}$

9. Is (2) a maximal ideal in $\mathbb{Z}[i]$? Justify your answer.

10. Show that the following ideals are prime ideal in the indicated ring
(a) I is set of all polynomials all of whose coefficients are even in $\mathbb{Z}[x]$.
(b) $I = \{f(x): f(0) = 0\}$ in $\mathbb{Z}[x]$. Also show that I is not maximal ideal.
(c) $I = \{(x, 0): x \in \mathbb{Z}\}$ in $\mathbb{Z} \times \mathbb{Z}$. Also show that I is not maximal ideal.
(d) $I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ divisible by } 5 \right\}$ in $M_2(\mathbb{Z})$
(e) $(x^3 + x + 1)$ in $Z_2[x]$
(f) $I = \{(3x, y): x, y \in Z\}$ in $Z \times Z$ under component wise addition and multiplication.
(g) $(x^2 + 1)$ in $\mathbb{Z}[x]$. Also show it is not maximal

11. Determine which of the following are prime ideals in $\mathbb{Z}[i]$?
(i) (2) (ii) (3) (iii) $(1 + i)$ (iv) $(2 + i)$

12. Show that $(2 + i)$ is a maximal ideal in $\mathbb{Z}[i]$. How many elements does $\mathbb{Z}[i]/(2 + i)$ have?

13. Consider the ring $\mathbb{Z}[\sqrt{d}]$, where d is not 1 and is not divisible by square of number. Define $N: \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}^+$ as $N(a + b\sqrt{d}) = |a^2 - db^2|$. Show that
(a) $N(x) = 0$ if and only if $x = 0$ (b) $N(xy) = N(x)N(y)$
(c) $N(x) = 1$ if and only if x is unit (d) x is irreducible if $N(x)$ is prime

14. In $\mathbb{Z}[\sqrt{5}]$, prove that 2 and $1 + \sqrt{5}$ are irreducible but not prime.

Paper II **Practical No.6** **Semester VI**

Polynomial Rings, Fields

Objective Questions

1. If $R_1 = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}, R_2 = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}, R_3 = \mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}, R_4 = \mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$, then
(a) R_1, R_2, R_3, R_4 are integral domains which are not fields.
(b) R_1, R_2, R_4 are integral domains which are not fields and R_3 is a field.
(c) R_1, R_2, R_3, R_4 are all fields
(d) None of the above
2. Consider the ring $S = \left\{\begin{pmatrix} a & a \\ a & a \end{pmatrix} a \in \mathbb{Q}\right\}$
(a) S is an integral domain which is not a field
(b) S is a field with multiplicative identity $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$
(c) S is a non-commutative ring
(d) None of these
3. The quotient ring $\frac{\mathbb{Z}[i]}{(1+i)}$
(a) an integral domain which is not a field (b) a field having 2 elements
(c) a field having 4 elements (d) a ring with proper zero divisors
4. The ring $\frac{\mathbb{R}[x]}{(x^4+1)}$ is
(a) an infinite integral domain (b) an infinite field
(c) a finite field (d) None of these
5. Let R and S be rings. Consider the ring $R \times S$ under component wise addition and multiplication.
(a) If R, S are fields, then $R \times S$ is a field
(b) If R, S are integral domains, then $R \times S$ is an integral domain
(c) $R \times S$ is not a field, whatever R, S may be.
(d) None of these
6. Let F_1 and F_2be fields having 9 and 16 elements respectively. Then, the number of (non-trivial) ring homomorphism from F_1 to F_2 are
(a) One (b) zero (c) two (d) None of the above.
7. $\frac{\mathbb{R}[x]}{(x^4+I)}$ is
(a) a finite field (b) an infinite integral domain which is not a field
(c) an infinite field (d) None of these
8. Consider the following rings
(i) $\mathbb{Z}_{10}$ (ii) $\mathbb{Z}_{17}$ (iii) $\left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0, b\ odd\right\}$ Then
(a) (i), (ii) and (iii) are fields. (b) (i), (ii) are fields
(c) (ii), (iii) are fields (d) only (ii) is a field
9. Let F be a field and R be a subring of F. Then
(a) R is a field (b) R is an integral domain but may not be a field

(c) R may not be an integral domain (d) R may not be commutative

10. There exists field of

(a) 10 elements (b) $7, 8, 9$ elements (c) 12 elements (d) 6 elements

11. Let $R = \mathbb{Z}[x]/_{(2x)}$ Then

(a) R is a field (b) R is an integral domain but not a field

(c) R is not an integral domain (d) R is a finite commutative ring

12. Let F_1, F_2 be fields, then $F_1 \times F_2$ under component wise addition and multiplication is

(a) a field (b) an integral domain which is not a field

(c) not an integral domain (d) a field iff F_1, F_2 are finite

13. The quotient field of quotients of $\mathbb{Z}[\sqrt{2}]$ is

(a) $\mathbb{Q}[\sqrt{2}]$ (b) $\mathbb{R}$ (c) $\mathbb{Q}$ (d) $\mathbb{C}$

14. The quotient field of quotient of $\mathbb{Z}[i]$ is

(a) $\mathbb{Q}[i]$ (b) $\mathbb{C}$ (c) $\mathbb{R}$ (d) None of the above

15. The field of quotients of $\mathbb{Q}[x]$ is

(a) $\mathbb{R}[x]$ (b) $\mathbb{C}[x]$ (c) $\mathbb{Q}(x) = \left\{\frac{f(x)}{g(x)} : f(x), g(x) \in Q[x]. g(x) \neq 0,\right\}$ (d) None of these

16. Which of the following statement is true?

(i) R is ring $\Rightarrow R[x]$ is ring (ii) R is division ring $\Rightarrow R[x]$ is division ring

(iii) R is field$\Rightarrow R[x]$ is field. (iv) R is integral domain $\Rightarrow R[x]$ is integral domain.

17. The polynomial $f(x) = 2x^2 + 4$ is reducible over

(a) $\mathbb{Z}$ (b) $\mathbb{Q}$ (c) $\mathbb{R}$ (d) None

18. Which of the following polynomials in $\mathbb{Z}[x]$ satisfy an Eisenstein criterion for irreducibility in $\mathbb{Q}$.

(i) $x^2 - 12$ (ii) $8x^3 = 6x^2 - 9x + 24$

(iii) $4x^{10} - 9x^3 + 24x - 18$ (iv) $2x^{10} - 25x^3 + 10x^2 - 30$

(a) All are irreducible (b) (ii) and (iii) are irreducible

(c) (ii), (iii) and (iv) are irreducible (d) only (i) is true

19. The polynomial $8x^3 - 6x + 1$ is

(a) reducible over $\mathbb{Z}$ (b) is reducible over $\mathbb{Q}$

(c) is irreducible over $\mathbb{Q}$ (d) is irreducible over $\mathbb{R}$

20. Let $f(x) \in \mathbb{Z}[x]$. Which of the following is true?

(a) If $f(x)$ is reducible over $\mathbb{Q}$, then it is reducible over $\mathbb{Z}$

(b) $f(x)$ is reducible over $\mathbb{Q}$, but it may not be reducible over $\mathbb{Z}$

(c) $f(x)$is reducible over $\mathbb{Q}$

(d) None of these

21. Let $f(x) = x^2 - 2$, then

(a) $f(x)$ is reducible in $\mathbb{Q}[x]$

(b) $f(x)$ is irreducible in $\mathbb{Q}[x]$ but reducible in $\mathbb{Q}[\sqrt{2}][x]$.

(c) $f(x0$ is reducible over $\mathbb{Q}$

(d) None of these

22. Let $f(x) = x^2 - 2$, then

(a) $f(x)$ is reducible in $\mathbb{Z}_3[x]$ and $\mathbb{Z}_5[x]$

(b) $f(x)$ is irreducible in $\mathbb{Z}_3[x]$ but reducible in $\mathbb{Z}_5[x]$.

(c) $f(x)$ is reducible in $\mathbb{Z}_3[x]$ but irreducible in $\mathbb{Z}_5[x]$

(d) $f(x)$ is irreducible in $\mathbb{Z}_3[x]$ as well as in $\mathbb{Z}_5[x]$.

23. Let R be a commutative ring and $f(x)$ be a polynomial of degree n over R.
Then the no of roots of $f(x)$ in R is

(a) less than or equal to n (b) equal to n

(c) strictly less than n (d) may be greater than n

24. The polynomial $2x + 1$ is

(a) unit in $\mathbb{Z}_8[x]$ (b) zero divisor in $\mathbb{Z}_8[x]$ but not nilpotent.

(c) nilpotent in $\mathbb{Z}_8[x]$ (d) None of the above

25. Let $I = (x^2 + x + 1)$ in $Z_n[x], 1 \leq n \leq 10$ Then, $\mathbb{Z}_n[x]/I$ is a field if

(a) $n \leq 5$ (b) $n = 2$ (c) $n = 3$ (d) $n = 7$

26. The polynomial x is irreducible in $\mathbb{Z}_n[x]$

(a) for each n (b) for $n \geq 3$ (c) f iff n is prime (d) not for all n

27. The polynomial $x^4 + 1$ is irreducible over

(a) $\mathbb{R}$ (b) $\mathbb{Q}$ (c) $\mathbb{C}$ (d) $\mathbb{Z}_2$

28. The number of roots of the polynomial $x^{25} - 1$ in $\mathbb{Z}_{37}$ is

(a) 25 (b) 5 (c) 24 (d) 1

29. Let $f(x) = x^3 - x^2 + 1$

(a) $\big(f(x)\big)$ is a maximal ideal in $\mathbb{Z}_2[x]$, $\mathbb{Z}_3[x]$ and $\mathbb{Z}_5[x]$

(b) $\big(f(x)\big)$ is a maximal ideal in $\mathbb{Z}_3[x]$ and $\mathbb{Z}_5[x]$ but not in $\mathbb{Z}_2[x]$

(c) $\big(f(x)\big)$ is a maximal ideal in $\mathbb{Z}_2[x] and$ $\mathbb{Z}_3[x]$ but not in $\mathbb{Z}_5[x]$

(d) None of the above

30. In the ring $\mathbb{Z}[x]$,

(a) (x) is a maximal ideal (b) (x) is a prime ideal which is not maximal

(c) there is no maximal ideal in $\mathbb{Z}[x]$ (d) (x) is not a prime ideal

31. Let $f(x) = x^{10} + x^9 + x^8 + \cdots + x + 1$
$g(x) = x^{11} + x^{10} + \cdots + x + 1$ Then

(a) $f(x), g(x)$ are both irreducible over $\mathbb{Z}[x]$.

(b) $f(x), g(x)$ are not irreducible over $\mathbb{Z}[x]$

(c) $f(x)$ is irreducible over $\mathbb{Z}[x], g[x]$ is not

(d) $g(x)$ is irreducible over $\mathbb{Z}[x], f(x)$ is not

32. Let R be a commutative ring. Then

(a) $\mathbb{R}[x]$ is an integral domain if R is an integral domain

(b) $\mathbb{R}[x]$ is a field if R is a field

(c) $\mathbb{R}[x]$ may not be commutative

(d) None of the above

33. The polynomial $f(x) = x$ is

(a) irreducible over any ring R

(b) irreducible but not prime over any ring R.

(c) can be factored in some polynomial ring.

(d) has no roots

34. $z[i]\big/(2+i)$ is

(a) a field having 3 elements (b) a field having 5 elements

(c) not an integral domain (d) ab infinite integral domain

35. In $\mathbb{R}[x]$, Let $I=\{f(x)\in\mathbb{R}[x]; f(2)=f'(2)=f''(2)=0\}$

$J=\{f(x)\in\mathbb{R}[x]: f(2)=0, f'(3)=0\}$

(a) I, J are ideals in $\mathbb{R}[x]$ (b) I is an ideal, J is not

(c) Neither I nor J is an ideal (d) I is a prime ideal in $\mathbb{R}[x]$

DESCRIPTIVE QUESTIONS

1. (a)Show that the rings $\mathbb{Q}\left[\sqrt{2}\right]=\left\{r+s\sqrt{2} : r,s \in Q\right\}$ and $Q[i]=\{r+s\,i : r\ s \in Q\}$ are fields.

   (b) Show that the rings $\mathbb{Z}\left[\sqrt{2}\right]$, $\mathbb{Z}[i]$ are integral domains but not fields, final their quotient fields.
2. Check if the fields $Q\left[\sqrt{2}\right]$ and $Q\left[\sqrt{5}\right]$ are isomorphic.
3. Show that $\mathbb{Z}[i]/_{(3)}$ is a field.
4. Prove that the ring $\mathbb{Z}_2[x]/(x^3+x+1)$ is a field, but $\mathbb{Z}_3[x]/(x^3+x+)$ is not a field.
5. Find all roots of the polynomial; x^2+3x+2 over $\mathbb{Z}[x]$.
6. List all polynomials of degree 2 over

   (a) $\mathbb{Z}_2[x]$ (b) $\mathbb{Z}_3[x]$
7. Show that (x) is not a maximal ideal in $\mathbb{Z}[x]$. Find all maximal ideals in $\mathbb{Z}[x]$.
8. Determine which of the given polynomials are irreducible over $\mathbb{Q}$.

   (i) $x^5+9x^4+12x^2+6$ (ii) x^4+3x^2+3 (iii) x^4+x+1

   (iv) x^5+5x^2+1 (v) $x^3-5x+10$ (vi) x^4-3x^2+9

   (vii) $2x^5-5x^4+5$ (viii) x^4+8
9. Find all monic polynomials of degree 2 over $\mathbb{Z}_5$.
10. Show that the polynomial x^4+1 is irreducible over $\mathbb{Z}_p[x]$. where p is prime $p>2$.
11. Show that $x^3+ax^2+bx+1 \in \mathbb{Z}[x]$ is reducible over $\mathbb{Z}$ iff either $a=b$ or $a+b=-2$.
12. Let $I=\{f(x)\in\mathbb{R}[x]: f(2)=f'(2)=f"(2)=0\}$ Show that I is a principal ideal in $\mathbb{R}[x]$ and find its generator.
13. Let $f(x)=x^{11}+x^{10}+\cdots+x^2+x+1$

    $g(x)=x^{10}+x^9+\cdots+x^2+x+1$

    Determine whether $f(x), g(x)$ are irreducible over $\mathbb{Q}$.
14. Show that x^n-p is irreducible over $\mathbb{Q}$ for each prime p.
15. Find all irreducible polynomials over $\mathbb{R}$ and $\mathbb{C}$.
16. Determine all ideals in $\frac{\mathbb{Z}[x]}{(2,x^3+1)}$ where $(2,x^3+1)=(2)+(x^3+1)$
17. Let $=\{a_0+a_1x+\cdots+a_nx^n : a_0\in\mathbb{Z}, a_i \in \mathbb{Q}\ for\ i\geq 1\}$. Show that R is an integral domain. Find units and primes in R. Is x a prime in R.

18. For any ring R, show that $\frac{R[x]}{\langle x\rangle} \simeq R$.

19. Show that $\frac{\mathbb{Q}[x]}{(x^3-2)}$ is a field

20. Let p be a prime. $p > 2$ Show that the number of irreducible quadratic polynomials of the form $x^2 + ax + b$ is $\frac{p(p-1)}{2}$.

21. Let p be a prime. Show that the number of irreducible polynomials in $\mathbb{Z}_p[x]$ is $\frac{p(p+1)}{2}$.

22. Show that $\frac{\mathbb{R}[x]}{(x^2+1)}$ is isomorphic to $\mathbb{C}$.

23. Let F be a field and let a be a non-zero element of F.
   (a) If $a\,f(x)$ is irreducible in $F[x]$, prove that $f(x)$ is irreducible in $F[x]$
   (b) If $f(ax)$ is irreducible in $F[x]$, prove that $f(x)$ is irreducible in $F[x]$
   (c) If $f(x+a)$ is irreducible in $F[x]$, prove that $f(x)$ is irreducible in $F[x]$.
   (d) use part (c) to prove that $8x^3 - 6x + 1$ is irreducible over $\mathbb{Q}$

24. If p is a prime, prove that $x^{p-1} - x^{p-2} + x^{p-3} \ldots - x + 1$ is irreducible over Q.

25. Let F be a field having 32 elements. Then show that the only sub field of F is $\{0, 1\}$ and F itself.

26. Show that $x^2 + 1$ and $x^2 + x + 4$ are irreducible polynomials in $\mathbb{Z}_{11}[x]$. Show that $\frac{\mathbb{Z}_{11}[x]}{(x^2+1)}$ and $\frac{\mathbb{Z}_{11}[x]}{(x^2+x+4)}$ are fields having 121 elements.

27. Construct a field of order (i) 25 (ii) 27

28. Show that a finite field containing p^n elements where p is a prime integer has characteristic p.

29. Suppose that $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$. If r is rational such that $x - r$ divides $f(x)$. Show that r is an integer.

30. Show that for every prime p there exists a field of order p^2.

Practical no 7. Unit-wise Theoretical Questions

Unit I

1. Let H be a subgroup of group G. Prove that the following statements are equivalent.
   (a) $aHa^{-1} \subseteq H$ for each $a \in G$.
   (b) $aHa^{-1} = H$ for each $a \in G$.
   (c) Every left coset of H in G is also a right coset of H in G i.e. $aH = Ha$ for each $a \in G$.
   (d) $HaHb = Hab$ for each $a, b \in G$.

2. Let G be a group. Show that centre of a G is a normal subgroup of G.

3. If H is a normal subgroup of G and K is a subgroup of G, Show that $HK = KH$.

4. Let G be a group and $a \in G$, Show that $N(a) = \{x \in G : ax = xa\}$ is a subgroup of G and $< a >$ is a normal subgroup of $N(a)$.

5. Let G be finite group with a normal subgroup H such that $(\circ(H), \circ(G/H)) = 1$ then show that H is a unique subgroup of G of order H.

6. Let G be a group and H is a unique subgroup of a given order, then show that H is a normal subgroup of G.

7. Let H and K be subgroup of a group G such that $H \cap K = \{e\}$ then show that $hk = kh,\ h \in H, k \in K$.

8. Let G be a group such that $(ab)^n = a^n b^n$ for some position integer n.
   (a) Show that $G(n) = \{x^n / x \in G\}$ is a normal subgroup of G.
   (b) Show that $G(n-1) = \{x^{n-1)} \,/\, x \in G\}$ is a normal subgroup of G.

9. Let H be a normal subgroup of G and let $\dfrac{G}{H} = \{Ha : a \in G\}$. Show that $HaHb = Hab$ is a well defined binary operation in $\dfrac{G}{H}$ and $\dfrac{G}{H}$ is a group under this binary operation.

10. Let G be a group and H be a subgroup of G. If $x^2 \in H$ for each $x \in G$, then show that H is a subgroup of G and G/H is Abelian.

11. If $G/Z(G)$ is cyclic then prove that G is Abelian.

12. If a cyclic subgroup H of a group G is normal in G. Show that every subgroup of H is normal in G .

13. Let G be a group and H be a normal subgroup of G. Then prove that
    (a) $(Ha)^n = Ha^n$ for all $n \in \mathbb{Z}$.
    (b) $\circ(Ha)$ divides $\circ(a)$.

14. Let G, G' be groups and $f : G \to G'$ be an onto homomorphism. Prove that

(a) kernel f is a normal subgroup of G and Imf is a subgroup of G'.

(b) If H' is a subgroup of G' then $f^{-1}(H') = \{h \in H : f(h) \in H'\}$ is a subgroup of G containing $\ker f$. If H' is normal in G' then $f^{-1}(H')$ is normal in G.

(c) If H is a subgroup G then $f(H) = \{f(h) : h \in H\}$ is a subgroup of G' and $f(Ha) = f(H)f(a)$ for each $a \in G$. Further, if H is normal in G then $f(H)$ is normal in G'.

15. Let G be a group and H be a normal subgroup of G. Show that $\eta : G \to G/H$ defined by $\eta(a) = Ha$ is a group homomorphism and $Ker\,\eta = H$.

16. State and prove "First isomorphism theorem / Fundamental theorem of homomorphism of groups".

17. State and prove " Second isomorphism theorem of groups".

18. State and prove "Third isomorphism theorem of groups".

19. State and prove Cayley's theorem for finite group.

20. Show that A_n is a normal subgroup of S_n.

21. Show that

(i) finite cyclic group of order n is isomorphic to the group $\mathbb{Z}_n$ of residue classes modulo n.

(ii) Every infinite cyclic group is isomorphic to the group $\mathbb{Z}$ of integers under addition.

OR

(i) Show that any two cyclic groups of same order are isomorphic.

(ii) Show that any two infinite cyclic groups are isomorphic.

22. Classify groups of order ≤ 7 up to isomorphism.

(i) Show the there are two non-isomorphic groups of order 4.

(ii) Show that there are only two non-isomorphic groups of order 6.

23. If $(G_1, \cdot), (G_2, *)$ are groups and $G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$ with binary operation $\circ$ defined by $(g_1, g_2) \circ (g_1', g_2') = (g_1 \cdot g_1', g_2 * g_2')$ then

(a) $(G_1 \times G_2, \circ)$ is a group.

(b) if G_1, G_2 are abelian then $G_1 \times G_2$ is also abelian.

(c) If $a \in G_1, b \in G_2$ such that $\circ(a) = m, \circ(b) = n$, then $(a, b)^k = (a^k, b^k)$ and $\circ(a, b) = lcm(m, n)$.

(d) If G_1, G_2 are cyclic then $G_1 \times G_2$ is cyclic if and only if $\circ(G_1)$ and $\circ(G_2)$ are relatively prime.

(e) If H_1, H_2 are normal subgroups of G_1, G_2 respectively then $H_1 \times H_2$ is a normal subgroup of $G_1 \times G_2$ and $\dfrac{G_1 \times G_2}{H_1 \times H_2}$ is isomorphic to the external direct product $\dfrac{G_1}{H_1} \times \dfrac{G_2}{H_2}$.

Unit II

1. R is a ring with multiplicative identity, then
   (a) Show that the set of units in R form a group under multiplication.
   (b) The set $Z(R) = \{a \in R \ : \ ax = xa; \ \forall\, x \in R\}$, called the center of the ring is a subring of R.

2. (a) Show that every element of a finite commutative ring is either a unit or a zero divisor.
   (b) Show that every element of $\mathbb{Z}_n$ is either a unit or a zero divisor.
   (c) Show that an integral domain has no non-zero nilpotent element.

3. Show that subring of an integral domain is an integral domain.

4. (a) Show that, characteristic of a ring R is n if and only if the order of the multiplicative identity of R is n in the group $(R, +)$.
   (b) Show that characteristic of an integral domain is either 0 or a prime.

5. (a) Let R be a ring with unity 1_R and I be an ideal in R such that $1_R \in I$ then prove that $I = R$.
   (b) Let R be a commutative ring and $a \in R$. Prove that $Ra = (a) = \{ra/r \in R\}$ is an ideal of R.
   (c) Show that any ideal of the ring $\mathbb{Z}$ is of the form $m\mathbb{Z}$ for some $m \in \mathbb{Z}$.

6. (a) If I is an ideal of a ring R, then show that $R/I = \{x + I \ : \ x \in R\}$ is a ring with the operations $(x + I) + (y + I) = (x + y) + I$ and $(x + I)(y + I) = xy + I$.
   (b) Let R be a commutative ring. If I, J are ideals in R, Show that $I \cap J$, $I + J$ and IJ are ideals of R, where

   $I + J = \{x + y \ : \ x \in I, y \in J\}$ and $IJ = \left\{ \sum_{i=1}^{n} x_i y_i \ : \ x_i \in I, y_i \in J, n \in \mathbb{N} \right\}$.

   (c) Let R be a ring and I, J, K be ideals of R. Prove (a) $I(J + K) = IJ + IK$, $(I + J)K = IJ + JK$. (b) If $J \subseteq I$, then $I \cap (J + K) = J + (I \cap K)$.
   (d) For a ring R, show that any ideal of the ring of $n \times n$ matrices over R, $M_n(R)$ is of the form $M_n(I) = \{[a_{ij}] \ : \ a_{ij} \in I\}$ for some ideal I of R.

7. Show that a commutative ring is a field if and only if it has no proper ideal.

8. Let I be an ideal in a ring R and $\eta : R \to R/I$ is defined by $\eta(a) = a + I$ for $a \in R$. Show that η is a homomorphism and $\ker \eta = I$.

9. Let R be a commutative ring. Show that $I = \{a : a \in R, a^n = 0 \text{ for some } n \in \mathbb{N}\}$ is an ideal (called the nil radical) of R and R/I has no nilpotent element.
10. Let R, R' be commutative rings and $f : R \to R'$ be a ring homomorphism. Show that-
    (a) If f is surjective, I is an ideal of R, then $f(I)$ is an ideal of R'.
    (b) If I' is an ideal of R' , then $f^{-1}(I')$ is an ideal of R.
11. State and prove the First Isomorphism Theorem(Fundamental theorem of homomorphism) of rings.
12. (Second Isomorphism Theorem of rings) Let A be a subring and B be an ideal of a ring R. Then $A \cap B$ is an ideal of A and $A/(A \cap B) \simeq (A + B)/B$.
13. (Third Isomorphism Theorem of rings) Let A, B be ideals of a ring R with $A \subseteq B$. Then A/B is an ideal of R/B and $(R/B)/(A/B) \simeq R/A$.
14. Show that, $\bar{J}$ is an ideal of the quotient ring R/I if and only if there is an ideal $J \subseteq I$ of the ring R such that $\bar{J} = \{x + I : x \in J\}$.
15. There is exactly one non-zero ring homomorphism from $\mathbb{Z}$ into any ring R.
16. Let $f : R \to S$ be an onto ring homomorphism and $K = ker\ f$. Prove that there is one-one onto correspondence between ideals of R containing K and ideals of S.

Unit III

Fields

1. Show that a field is an integral domain. Is the converse true? Justify your answer.
2. Show that a finite integral domain is a field. Give an example of an infinite integral domain which is not a field.
3. Show that characteristic of a field is either zero or a prime number.
4. Show that the ring $\mathbb{Z}_n$ of residue classes modulo n is field if and only if n is a prime number.
5. Show that a field has no ideals except 0 and itself.
6. Show that an ideal P in a commutative ring R is a prime ideal if and only if R/P is an integral domain.
7. Show that an ideal M in a commutative ring R is a maximal ideal if and only if R/M is a field.
8. (a) If R is a finite commutative ring prove that every prime ideal is maximal.
    (b) If R is a commutative ring such that for $a \in R$ there exists a $n \in \mathbb{N}$ (depending on a) such that $a^n = a$ then show that every prime ideal is maximal.
9. (a) Show that an ideal I in the ring $\mathbb{Z}$ of integers is a prime ideal if and only if $I = (0)$ or $I = p\mathbb{Z}$ where p is a prime number.

(b) Show that every non-zero prime ideal in $\mathbb{Z}$ is a maximal ideal.

(c) Show that an ideal I in the ring $\mathbb{Z}$ of integers is a maximal ideal if and only if $I = p\mathbb{Z}$ where p is a prime number.

10. Show that a field contains a subfield isomorphic to $\mathbb{Z}_p$ or $\mathbb{Q}$.

11. Explain construction of quotient field of $\mathbb{Z}$.

12. Show that the rings, $\mathbb{Z}[i], \mathbb{Z}[\sqrt{2}], \mathbb{Z}[\sqrt{-5}]$ are integral domain which are not fields. Show that their quotient fields are $\mathbb{Q}[i], \mathbb{Q}[\sqrt{2}], \mathbb{Q}[\sqrt{-5}]$ respective.

Polynomial Rings

1. Let R be a ring. Let $R[x] = \{a_n x^x + a_{n-1}x^{n-1} + \cdots a_1 x + a_0 : \ a_i \in R, n \in \mathbb{Z}^+\}$. Show that $R[x]$ is a ring with respect to usual addition and multiplication of polynomial. Further show that if R is an integral domain, then $R[x]$ is also an integral domain.

2. Let $\mathbb{F}$ be a field.

   (a) Show that $\mathbb{F}[x]$ is an integral domain. Is it a field? Justify your answer.

   (b) Show that only units in $\mathbb{F}[x]$ are the non-zero elements of $\mathbb{F}$.

   (c) Division Algorithm: For any pair of non-constant polynomials $f(x), g(x) \in \mathbb{F}[x]$, there exist $q(x), r(x) \in \mathbb{F}[x]$ such that $f(x) = g(x)q(x) + r(x)$ where $r(x) = 0$ or $deg\, r(x) < deg\, g(x)$.

3. Let F be a field. Show that every ideal of $F[x]$ is principal ideal.

4. Let F be a field $a \in F$, and $f(x) \in F[x]$. Then a is a zero of $f(x)$ if and only $x - a$ is a factor of $f(x)$.

5. Define irreducible polynomials. Let F be a field, $f(x) \in F[x]$ and deg $f(x) = 2$ or 3. Show that $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F.

6. Show that

   (a) if F is a field, $f(x)$ and $g(x)$ in $F[x]$ are associate if and only if $f(x) = cg(x)$ where $c \neq 0$ in R.

   OR

   if R is an integral domain $f(x)$ and $g(x)$ in $R[x]$ are associate iff $f(x) = cg(x)$ where c is a unit in R.

   (b) Let F be a field and let $f(x), g(x), h(x) \in F[x]$. If $f(x)$ is irreducible over F and $f(x)|g(x)h(x)$,then $f(x)|g(x)$ or $f(x)|h(x)$.
   (In $\mathbb{R}[x]$ or $\mathbb{Q}[x], \mathbb{C}[x]$) if f(x) is irreducible and $f(x)|g(x)h(x)$,then $f(x)|g(x)$ or $f(x)|h(x)$.

7. Let $\mathbb{F}$ be a field. Show that $(p(x))$ is a maximal ideal in $f[x]$ if and only if $p(x)$ is an irreducible polynomial in $F[x]$.

OR

Let F be a field. Show that $F[x]/ < p(x) >$ is a field if and only if $p(x)$ is an irreducible polynomial in $F[x]$.

8. Show that any a non-zero ideal of $\mathbb{F}[x]$ is prime if and only if it is maximal.

9. Show that the only irreducible polynomials in $\mathbb{R}[x]$ are a linear polynomial $x-a$ or quadratic polynomial $x^2 + bx + c$ such that $b^2 - 4c < 0$, where $a, b, c \in \mathbb{R}$.

OR

Show that the only maximal (or prime)ideals in $\mathbb{R}[x]$ are principal ideals $< x - a >$ or $< x^2 + bx + c >$ such that $b^2 - ac < 0$, $a, b, c \in \mathbb{R}$.

10. Show that the only irreducible polynomials in $\mathbb{C}[x]$ are a linear polynomial $x-\alpha$ for $\alpha \in \mathbb{C}$.

OR

Show that the only maximal (or prime)ideals in $\mathbb{C}[x]$ are principal ideals $< x - \alpha >$ where $\alpha \in \mathbb{C}$.

11. **Eisenstein's Criteria for Irreducibility** Let $f(x) = x^n + +a_{n-1}x^{n-1} + \cdots a_1x + a_0 \in \mathbb{Z}[x]$. Let $p \in \mathbb{Z}$ be a prime such that $p \mid a_i$, for all $i = 1, 2, \cdots n-1$ and $p^2 \nmid a_0$. Then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

12. Using Eisenstein's criteria show that the $p^{th}-$ Cyclotomic polynomial $\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$ where p is prime, is irreducible over $\mathbb{Q}$.

13. Let $f(x) = a_nx^n + \cdots + a_0 \in \mathbb{Z}[x]$ and $a_n \neq 0$ if $r/s \in \mathbb{Q}$,$(r, s) = 1$, $f(r/s) = 0$ then show that $r/a_n, s/a_0$.

14. Show that $p^{1/n}$ is irrational where $n > 1$ and p is a prime.

Divisibility

1. Let R be a commutative ring and $a, b, u \neq 0$. Then show that
   (a) If u is an unit in R then $u|a$.
   (b) $b \in (a) \Leftrightarrow a|b \leftrightarrow (b) \subseteq (a)$.
   (c) a and b are associates $\leftrightarrow$ $(a) = (b)$
   (d) If $a|1_R \leftrightarrow a$ is a unit and $R = (a)$.

2. Let R be an Integral Domain , Let $p \in R$. Then ,
   (a) p is prime iff (p) is a non zero prime ideal of R.
   (b) If p is prime then p is irreducible. Show that the converse is not true.

3. Prove that in $\mathbb{Z}$ (ring of integers) a non zero non unit element p is irreducible iff p is prime.

4. Let R be an Integral Domain and $a \in R$, $a \neq 0_R$. If (a) is maximal then a is irreducible. Give an example to show that converse is not true.

5. Let R be a commutative ring and I, J be prime ideals of R. Show that, $I \cap J$ is prime only if $I \subseteq J$ or $J \subseteq I$.
6. Let R be commutative and I, J be ideal of R and P is a prime ideal of R that contains $I \cap J$. Prove that either $I \subseteq P$ or $J \subseteq P$.
7. Let p be a non-zero element in an integral domain R. Then, if p is irreducible then $R/(p)$ is a field and (p) is a maximal ideal.

Topology of Metric Spaces and Real Analysis: Practical 3.1
Continuous Functions on Metric Spaces.
Objective Questions 3.1

. (Revised Syllabus 2018-19)

(1) Let d be the usual distance in $\mathbb{R}$. For any $A \subseteq \mathbb{R}, d_A : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $d_A(x) = \inf\{d(x,a) : a \in A\}$. Then
(a) $d_{\mathbb{R}}, d_{\mathbb{R}\setminus\mathbb{Q}}$ are not continuous on $\mathbb{R}$ and $d_{\mathbb{Q}}(x) > 0 \quad \forall x \in \mathbb{R}\setminus\mathbb{Q}$.
(b) $d_{\mathbb{Q}} \equiv 0$ and $d_{\mathbb{R}\setminus\mathbb{Q}} \equiv 0$ on $\mathbb{R}$ and $d_{\mathbb{Q}}, d_{\mathbb{R}\setminus\mathbb{Q}}$ are continuous on $\mathbb{R}$.
(c) $d_{\mathbb{R}}, d_{\mathbb{R}\setminus\mathbb{Q}}$ are continuous on $\mathbb{R}$ and $d_{\mathbb{R}\setminus\mathbb{Q}}(x) > 0 \quad \forall x \in \mathbb{Q}$.
(d) None of the above.

(2) Let d denote the usual distance in $\mathbb{R}$ and for $A \subseteq \mathbb{R}$, let

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} .$$

Then
(a) χ_A is continuous on $\mathbb{R}$ if and only if A is an open subset of $\mathbb{R}$.
(b) χ_A is continuous on $\mathbb{R}$ if and only if A is a closed subset of $\mathbb{R}$.
(c) χ_A is continuous on $\mathbb{R}$ if and only if $A = \emptyset$ or $A = \mathbb{R}$.
(d) None of the above.

(3) Consider the metrics d and d_1 on $\mathbb{N}$, where d is the induced distance from $\mathbb{R}$ with usual distance and $d_1(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right|$ for $m, n \in \mathbb{N}$. Let $i : \mathbb{N} \longrightarrow \mathbb{N}$ denote the identity map on $\mathbb{N}$. Then
(a) $i : (\mathbb{N}, d) \longrightarrow (\mathbb{N}, d_1)$ is continuous but $i : (\mathbb{N}, d_1) \longrightarrow (\mathbb{N}, d)$ is not continuous.
(b) $i : (\mathbb{N}, d) \longrightarrow (\mathbb{N}, d_1)$ is not continuous.
(c) $i : (\mathbb{N}, d_1) \longrightarrow (\mathbb{N}, d)$ is not continuous.
(d) None of the above.

(4) Let d_1 and d_2 be equivalent metrics on X and (Y,d) be any metric space. If $f : (X, d_1) \longrightarrow (Y,d)$ and $g : (Y,d) \longrightarrow (X, d_1)$ are continuous maps on X and Y respectively, then
(a) $f : (X, d_2) \longrightarrow (Y,d)$ is continuous, but $g : (Y,d) \longrightarrow (X,d_2)$ may not be continuous
(b) $f : (X, d_2) \longrightarrow (Y,d)$ may not be continuous, but $g : (Y,d) \longrightarrow (X,d_2)$ is continuous
(c) $f : (X, d_2) \longrightarrow (Y,d)$ and $g : (Y,d) \longrightarrow (X,d_2)$ are continuous on X and Y respectively.
(d) None of the above.

(5) Let $A = \{x \in \mathbb{R} : \sin x = \frac{1}{2}\}$, the distance in $\mathbb{R}$ being usual. Then
(a) A is an infinite closed set. (b) A is a finite closed set.
(c) A is an open set. (d) None of the above.

(6) Let (X,d) and (Y,d') be metric spaces and $f, g : X \longrightarrow Y$ be continuous maps. If $A \subseteq X$ such that $f(x) = g(x) \quad \forall\, x \in A$, then the statement which is **not true** is

(a) $f(x) = g(x) \quad \forall x \in A^\circ$
(b) $f(x) = g(x) \quad \forall\, x \in \overline{A}$
(c) $f(x) = g(x) \quad \forall\, x \in \delta A$ where δA is the boundary of A
(d) All the above statements are false

(7) Let (X, d) and (Y, d') be metric spaces and $f, g : X \longrightarrow Y$ be continuous maps. Let $A = \{x \in X : f(x) = g(x)\}$. Then
(a) A is a dense subset of X. (b) A is a closed subset of X.
(c) A is an open subset of X. (d) None of the above.

(8) Let d denote the usual distance in $\mathbb{R}$ and d_1 denote the discrete metric on $\mathbb{R}$. Let $i : (\mathbb{R}, d_1) \longrightarrow (\mathbb{R}, d)$ be the identity map. Then
(a) $\overline{i(\mathbb{Q})} \subseteq i(\overline{\mathbb{Q}})$. (b) $i^{-1}(\overline{\mathbb{Q}}) \subseteq \overline{i^{-1}(\mathbb{Q})}$.
(c) $\overline{i^{-1}(\mathbb{Q})} \subseteq i^{-1}(\overline{\mathbb{Q}})$. (d) None of the above.

(9) Let (X, d) and (Y, d') be metric spaces and $f : X \longrightarrow Y$. Let $\{A_n\}_{n\in\mathbb{N}}$ be a family of closed subsets of X. Then the statement which is **not true** is
(a) If f is continuous on A_1 and A_2, then f is continuous on $A_1 \cup A_2$.
(b) If f is continuous on each A_n, then f is continuous on $\displaystyle\bigcup_{n=1}^{\infty} A_n$.
(c) If f is continuous on each A_n, then f is continuous on $A = \displaystyle\bigcap_{n\in\mathbb{N}} A_n$, provided $A \neq \emptyset$
(d) None of the above.

(10) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (the distance in $\mathbb{R}$ and $\mathbb{R}^2$ are Euclidean) be defined by $f(x, y) = |x|$. Then
(a) f is not continuous at $(x, 0)$ for each $x \in \mathbb{Z}$.
(b) f is not continuous at $(0, 0)$.
(c) f is continuous on $\mathbb{R}^2$
(d) None of the above.

(11) $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are any maps, such that $f \circ g$ and $g \circ f$ are continuous (distance being usual). Then
(a) $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous
(b) $f \circ g = g \circ f$
(c) At least one of f and g is coninuous.
(d) Neither f nor g may be continuous.

(12) Let (X, d) be a metric space where X is a finite set and (Y, d') be any metric space. Let $f : X \longrightarrow Y$. Then the statement which is **not true** is
(a) f is continuous on X
(b) $f(X)$ is bounded.
(c) If A is open in X, $f(A)$ is open in Y
(d) If B is closed in $Y, f^{-1}(B)$ is closed in X.

(13) Let (X, d) be a compact metric space and $f : X \longrightarrow (0, \infty)$. (distance usual) be a continuous function. If $\inf\{f(x) : x \in X\} = m$, then
(a) m may be 0 (b) $m = 0$ (c) $m > 0$ (d) m may be negative

(14) Let (X, d) be a finite metric space, $|X| > 2$. If $f : X \longrightarrow \mathbb{R}$ (usual distance) is a continuous function, then
(a) $|f(X)| \geq 2$ (b) $f(X) = [m, M]$ for some $m, M \in \mathbb{R}$
(c) f is a constant function. (d) None of the above.

(15) Let (X, d) be a metric space. If $f, g \in C(X, \mathbb{R})$, then
(a) $f + g \in C(X, \mathbb{R})$, but $f - g$ may not be in $C(X, \mathbb{R})$.
(b) $f + g, f - g$ and $2f \in C(X, \mathbb{R})$.
(c) $f + g, f - g \in C(X, \mathbb{R})$, but $2f$ may not be in $C(X, \mathbb{R})$
(d) $f + g, f - g \in C(X, \mathbb{R})$, but fg may not be in $C(X, \mathbb{R})$

(16) Let $X_1 = [0, 1]; Y_1 = [0, \infty); X_2 = (0, 1) \cup (2, 3), Y_2 = (0, 1); X_3 = (0, 1), Y_3 = \{0, 1\}$. Then there exists a continuous onto function from $X_i \longrightarrow Y_i$ when
(a) $i = 1, 2, 3$ (b) $i = 1, 2$ (c) $i = 2$ (d) $i = 3$

(17) Consider the map $L : C[0, 1] \longrightarrow \mathbb{R}$ (usual distance) defined by $L(f) = \int_0^1 f(t)\, dt$. Then,
(a) $L : (C[0, 1], \| \|_1) \longrightarrow \mathbb{R}$ is continuous but $L : (C[0, 1], \| \|_\infty) \longrightarrow \mathbb{R}$ is not continuous.
(b) $L : (C[0, 1], \| \|_\infty) \longrightarrow \mathbb{R}$ is not continuous.
(c) $L : (C[0, 1], \| \|_1) \longrightarrow \mathbb{R}$ and $L : (C[0, 1], \| \|_\infty) \longrightarrow \mathbb{R}$ are both not continuous.
(d) None of the above.

(18) Consider the map $\phi : C[0, 1] \longrightarrow \mathbb{R}$ defined by $\phi(f) = f(0)$. Then
(a) $\phi : (C[0, 1], \| \|_\infty) \longrightarrow \mathbb{R}$ is not continuous.
(b) $\phi : (C[0, 1], \| \|_\infty) \longrightarrow \mathbb{R}$ is continuous.
(c) $\phi : (C[0, 1], \| \|_\infty) \longrightarrow \mathbb{R}$ and $\phi : (C[0, 1], \| \|_1) \longrightarrow \mathbb{R}$ are not continuous.
(d) None of the above.

(19) Let (X, d) be a metric space and $f \in C(X, \mathbb{R})$ be a bounded function. Then f

(a) attains both bounds.
(b) attains at least one bound.
(c) may not attain either bound.
(d) None of the above.

(20) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous function (distance usual) and $A \subseteq \mathbb{R}$. Consider the following statements:
(i) If A is closed and bounded, $f(A)$ is closed and bounded.
(ii) If A is closed, $f(A)$ is closed. (iii) If A is bounded, $f(A)$ is bounded.
(a) (i), (ii), (iii) are true statements.
(b) (i) and (iii) are true, (ii) is not true.
(c) Only (i) is true.
(d) (i) and (ii) are true, (iii) is not true.

(21) Let (X, d) be a compact metric space and $f : X \longrightarrow \mathbb{R}$ is continuous. Let (x_n) be a sequence in X. Which statement is false?
(a) If (x_n) is convergent $(f(x_n))$ is convergent.
(b) If (x_n) is Cauchy, $(f(x_n))$ is Cauchy.
(c) $(f(x_n))$ has convergent subsequence.
(d) None of (a), (b), (c) are false.

Topology of Metric Spaces and Real Analysis: Practical 3.1
Continuous Functions on Metric Spaces
Descriptive Questions 3.1

(1) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous function (with respect to usual distance). Let $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $h(x, y) = (f(x), g(y))$. Show that $h : (\mathbb{R}^2, d) \longrightarrow (\mathbb{R}^2, d)$ is Continuous where d is Euclidean distance.

(2) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be continuous map. Show that $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $g(x, y) = f(x + y, x - y)$ is continuous.

(3) Show that $i : (\mathbb{R}, d) \longrightarrow (\mathbb{R}, d_1)$ where d is usual distance in $\mathbb{R}$ and d_1 is discrete metric on $\mathbb{R}$ is not continuous where i is the identity map on $\mathbb{R}$.

(4) Let (X, d) be a metric space and let $A \subseteq X$, If $d_A : X \longrightarrow \mathbb{R}$ is defined by $d_A(x) = d(x, A)$. Show that d_A is continuous.

(5) $X = M_2(\mathbb{R})$ and $\|A\| = \left(\sum_{1 \le i,j \le 2} a^2_{(ij)}\right)^{\frac{1}{2}}$. Show that $f : X \longrightarrow \mathbb{R}$ (distance usual) defined by $f(A) =$ det$A$ is continuous. Hence show that
(i) $(GL)_2(\mathbb{R})$ is an open subset of X.
ii) $(SL)_2(\mathbb{R})$ is a closed subset of X.

(6) Prove or disprove:
a) If (X, d) and (Y, d') are metric spaces and $f : X \longrightarrow Y$ is a continuous bijective map, then for any open ball B in (X, d), $f(B)$ is an open ball (Y, d').
b) Let (X, d) and (Y, d') be metric spaces. If (X, d) is complete and $f : X \longrightarrow Y$ is continuous and onto, then (Y, d') is complete.

(7) Let (X, d) and (Y, d') be metric spaces. Prove that $f : X \longrightarrow Y$ is continuous on X if and only if f is continuous on each compact subset of X.

(8) Let A, B be two compact subsets of a metric space (X, d) such that $A \cap B \neq \emptyset$. Show that $d(A, B) > 0$ and $\exists\ a \in A, b \in B$ such that $d(A, B) = d(a, b)$.

(9) Let $K \subseteq \mathbb{R}^n$ be such that any continuous function from$K$ to $\mathbb{R}$ be bounded. Show that K is compact.

(10) Show that $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a compact subset of $\mathbb{R}^2$, distance being Euclidean.

(11) Let $f : X \longrightarrow (0, \infty)$ be a continuous function, where (X, d) is a compact metric space. Show that $\exists\ \epsilon > 0$ such that $f(x) \geq \epsilon, \forall x \in X$.

(12) $\psi : (C[0, 1], \| \ \|_\infty) \longrightarrow \mathbb{R}$ (usual distance) defined by $\psi(f) = f(0)$ is continuous.

(13) $L : (C[0, 1], \| \ \|_\infty) \longrightarrow \mathbb{R}$ (usual distance) defined by $L(f) = \displaystyle\int_0^1 f(t)\ dt$ is continuous.

Topology of Metric Spaces and Real Analysis: Practical 3.2
Uniform Continuity and Fixed Point Theorem
Objective Questions 3.2

. (Revised Syllabus 2018-19)

(1) $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ defined by, $f(x) = \dfrac{1}{x}$ for $x \neq 0$ is uniformly continuous on
(a) $(0, 1)$ (b) $(0, \infty)$ (c) $[1, \infty)$ (d) None of these.

(2) $f(x) = \dfrac{1}{1 + x^2}$ for $x \in \mathbb{R}$ is uniformly continuous on
(a) $[0, 1]$ but not on $[0, \infty)$
(b) $[1, \infty)$ but not on $[0, \infty)$
(c) $\mathbb{R}$
(d) None of these.

(3) Let $A \subseteq \mathbb{R}$. If $f, g : A \longrightarrow \mathbb{R}$ are uniformly continuous on A, then
(a) $f + g$ is uniformly continuous on A but $f \cdot g$ may not be uniformly continuous on A.
(b) $f + g$ and $f \cdot g$ are uniformly continuous on A.
(c) Neither $f + g$ nor $f \cdot g$ may be uniformly continuous on A.
(d) None of the above.

(4) Consider the following functions (distance in $\mathbb{R}$ is usual):

(i) $f : [0, 2\pi] \longrightarrow \mathbb{R}, f(x) = x \sin x$
(ii) $f : (0, 1) \longrightarrow \mathbb{R}, f(x) = \frac{1}{x}$
(iii) $f : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}, f(x, y) = x + y$ (distance in $\mathbb{R}^2$ Euclidean)

(a) (i), (ii), (iii) are uniformly continuous.
(b) (i) and (iii) are uniformly continuous, (ii) is not.
(c) Only (i) is uniformly continuous.
(d) Only (iii) is uniformly continuous.

(5) Suppose A and B are closed subsets of $\mathbb{R}$ and $f : A \cup B \longrightarrow \mathbb{R}$ is uniformly continuous on A as well as B. Then,
(a) f is uniformly continuous on $A \cup B$.
(b) f is uniformly continuous on $A \cup B$ if $A \cap B = \emptyset$.
(c) f may not be uniformly continuous on $A \cup B$.
(d) None of the above.

(6) $f : [0, \infty) \longrightarrow \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is
(a) continuous on $[0, \infty)$ but not uniformly continuous on $[0, \infty)$.
(b) uniformly continuous on $[0, 1]$ but not on $[0, \infty)$.
(c) uniformly continuous on $[0, \infty)$.
(d) None of the above.

(7) If $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are uniformly continuous on $\mathbb{R}$, then
(a) The product $f \cdot g$ uniformly continuous on $\mathbb{R}$.
(b) The composites $f \circ g$ and $g \circ f$ uniformly continuous on $\mathbb{R}$.
(c) f^2 and g^2 are uniformly continuous on $\mathbb{R}$.
(d) None of the above.

(8) Let (X, d) be a metric space and A be a non-empty subset of X. Then $d_A : X \longrightarrow \mathbb{R}$ defined by $d_A(x) = d(x, A) = \inf\{d(x, a) : a \in A\}$ is

(a) continuous on A but not on X.
(b) uniformly continuous on X.
(c) not uniformly continuous on X.
(d) None of these.

(9) Let (X, d) and (Y, d') be a metric spaces and $f : X \longrightarrow Y$. Suppose (x_n) is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y if

(a) f is continuous on X.
(b) f is uniformly continuous on X.
(c) X and Y are complete.
(d) None of these.

(10) Let $A \subseteq \mathbb{R}, A$ is bounded but not closed. Then
(a) Any continuous function from A to $\mathbb{R}$ is bounded.
(b) Any continuous function from A to $\mathbb{R}$ is uniformly continuous.
(c) Any continuous, bounded function from A to $\mathbb{R}$ attains bounds.
(d) None of the above.

(11) Let (X, d) and (Y, d') be metric spaces and $f : X \longrightarrow Y$ a uniformly continuous function. Then the statement which is not true is
(a) Given a bounded subset A of X, $f(A)$ need not be a bounded subset of Y.
(b) If $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y.
(c) If $\{f(x_n)\}$ is a Cauchy sequence in Y, $\{x_n\}$ is a Cauchy sequence in X.
(d) If $\{x_n\}$ is convergent, then $\{f(x_n)\}$ is convergent.

(12) Consider the following maps:

(i) $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that f is differentiable and $|f'(x)| \leq M \ \forall \ x \in \mathbb{R}$.

(ii) A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

(iii) A map $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying Lipchitz condition namely $\exists \ M \geq 0$ such that $|f(x) - f(y)| \leq M|x - y| \quad \forall \ x, y \in \ \mathbb{R}$.

Then
(a) (i) and (iii) are uniformly continuous.
(b) (i), (ii) and (iii) are uniformly continuous.
(c) only (iii) is uniformly continuous.
(d) None of above.

(13) Which of the following real valued functions are uniformly continuous on the give sets.

(i) $f(x) = \dfrac{1}{x}$ on $(0, 1)$. (ii) $f(x) = x^{\frac{1}{3}}$ on $[0, 1]$.

(a) Only (i) (b) only (ii) (c) both (i) and (ii) (d) Neither (i) nor (ii).

Topology of Metric Spaces and Real Analysis: Practical 3.2
Uniform Continuity and Fixed Point Theorem
Descriptive Questions 3.2

(1) Show that the function $f(x) = \dfrac{1}{1+x^2}$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

(2) Prove or disprove:
If $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are uniformly continuous on a nonempty set $A \subseteq \mathbb{R}$ then the product function $f \cdot g$ is uniformly continuous on A.

(3) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is such that $f'(x)$ exists $\forall\ x \in\ \mathbb{R}$ and $\exists$ a constant M such that $|f'(x)| \leq M \quad \forall\ x \in\ \mathbb{R}$, then show that f is uniformly continuous on $\mathbb{R}$.

(4) If $(X, d), (Y, d')$ are metric spaces, then prove that any Lipschitz function $f : (X, d) \longrightarrow (Y, d)$ is uniformly continuous. Hence, deduce that $\sin x, \cos x$ are uniformly continuous on $\mathbb{R}$.

(5) Let $A = (0, 1] \subset \mathbb{R}$. Define $d_A : \mathring{A} \to \mathbb{R}$ as $d_A(x) = d(x, A)$. Draw graph of d_A. Further, prove that if (X, d) is a metric space and $A \subseteq X$ then $d_A : X \longrightarrow \mathbb{R}$ defined by $d_A(x) = d(x, A)$ for $x \in\ X$ is uniformly continuous on X.

(6) Let $f : [a, b] \longrightarrow [a, b]$ be differentiable and $|f'(x)| \leq c$ with $0 < c < 1$. Then show that f is a contraction of $[a, b]$.

(7) Let X and Y be metric spaces. Assume that Y is a discrete metric space and that $f : X \longrightarrow Y$ is a contraction. What can you conclude about f?

(8) Define a sequence of positive real numbers by letting x_0 to be any positive real number and $x_{n+1} = (1 + x_n)^{-1}$. Show that this sequence converges and find its limit. (Hint: Prove that f is a contraction mapping where $f : [x_0, \infty) \to \mathbb{R}$ defined as $f(x) = \dfrac{1}{1+x}$).

Topology of Metric Spaces and Real Analysis: Practical 3.3
Connected Sets , Connected Metric Spaces
Objective Questions 3.3

. (Revised Syllabus 2018-19)

(1) Let (X, d) be a discrete metric space
   (a) X is connected.
   (b) X is connected only if X is infinite.
   (c) X is connected if and only if X is a singleton set.
   (d) None of above.

(2) Let d be usual distance in $\mathbb{R}$ and d_1 be the discrete metric in $\mathbb{R}$. Then
   (a) $[0, 1]$ is a connected subset $(\mathbb{R}, d)$ as well as $(\mathbb{R}, d_1)$.
   (b) $[0, 1]$ is connected subset of $(\mathbb{R}, d)$ but not connected subset of $(\mathbb{R}, d_1)$.
   (c) $[0, 1]$ is not a connected subset of $(\mathbb{R}, d)$ but a connected subset of $(\mathbb{R}, d_1)$
   (d) $[0, 1]$ is not a connected subset of $(\mathbb{R}, d)$ as well as $(\mathbb{R}, d_1)$

(3) If A is a connected subset of $(\mathbb{R}, d)$ (d being usual distance) then
   (a) A° and $\overline{A}$ are connected.
   (b) A° may not be connected but $\overline{A}$ is connected.
   (c) Both A° and $\overline{A}$ may not be connected.
   (d) A° is connected, but $\overline{A}$ may not be connected.

(4) Let A, B be connected subsets of $(\mathbb{R}, d)$ where d is the usual distance in $\mathbb{R}$. If $A \cap B \neq \emptyset$, then the following set may not be connected.
   (a) $A \cup B$ (b) $A \cap B$ (c) $A \setminus B$ (d) $A \times B$ in $\mathbb{R}^2$ (Euclidean distance).

(5) Let $A \subseteq \mathbb{Q}$. If A is a connected subset of $(\mathbb{R}, d)$ where d is usual distance then
   (a) $A = \mathbb{Q}$ (b) A is an infinite bounded set.
   (c) A is a singleton set. (d) None of the above.

(6) Consider the following subsets of $(\mathbb{R}^2, d)$ where d Euclidean .
   $(i)\{(x, y) \in \mathbb{R}^2 : xy = 1\}$ $(ii)\{(x, y) \in \mathbb{R}^2 : x = 0\}$
   $(iii)\{(x, y) \in \mathbb{R}^2 : xy = 0\}$ Then,
   (a) $(i), (ii)(iii)$ are all connected. (b) $(ii), (iii)$ are connected.
   (c) Only (iii) is connected. (d) Only (i) is connected.

(7) Let A, B be non-empty closed subsets of a metric space (X, d). If $A \cup B$ and $A \cap B$ are connected subsets of X, Then6 (a) A and B are both connected.
   (b) A and B are both not connected.
   (c) A and B are connected if and only if $A = B$
   (d) None of these.

(8) Let (X, d) be a finite metric space. If $A \subseteq X$ is connected then
   (a) $A = X$ (b) $A \neq X$ (c) A is a singleton set. (d) A has more than one element.

(9) If A, B are connected subsets of $(\mathbb{R}^2, d)$ where d is usual distance and $A \cap B \neq \emptyset$, then
(a) $A \cup B$ is connected but $A \cap B$ may not be connected.
(b) $A \cup B$ may not be connected but $A \cap B$ is connected.
(c) $A \cup B$ and $A \cap B$ are connected.
(d) None of the above.

(10) Consider $(\mathbb{R}^2, d)$ where d is Euclidean metric and A be an open ball in $\mathbb{R}^2$ and L be a line in $\mathbb{R}^2$. Then
(a) $A \cup L$ is connected if L does not intersect A.
(b) $A \cup L$ is connected if L intersects A.
(c) $A \cup L$ is disconnected if L intersects A but is not a tangent to A.
(d) Cannot say.

(11) In $(\mathbb{R}^2, d)$ where d is Euclidean distance, the following set is not connected.
(a) $\mathbb{R}^2 \setminus \mathbb{Q} \times \mathbb{Q}$.
(b) $\mathbb{R}^2 \setminus \{(0, 0)\}$
(c) $\mathbb{R}^2 \setminus \{(x, y) : y = 0\}$
(d) None of the above.

(12) If A, B are connected subsets of $(\mathbb{R}, d)$ where d is usual and $A \cap B \neq \emptyset$, then
(a) $A \cup B$ is connected but $A \cap B$ may not be connected.
(b) $A \cup B$ may not be connected but $A \cap B$ is connected.
(c) $A \cup B$ and $A \cap B$ are connected.
(d) None of the above.

(13) Let A and B be connected subsets in a metric space (X, d) and $A \subseteq C \subseteq B$ Then,
(a) C is connected .
(b) C° is connected.
(c) $\overline{C}$ is connected.
(d) $C \cap \overline{A}$ is connected.

Topology of Metric Spaces and Real Analysis: Practical 3.3
Connected Sets , Connected Metric Spaces
Descriptive Questions 3.3

(1) Let (X, d) be a metric space and $A, B \subseteq X$ be closed. Prove that $A \cap B^c$ and $B \cap A^c$ separated.

(2) Let (X, d) be a metric space and $A, B, C \subseteq X$. If A and B are separated, B and C are separated, then prove that $A \cup C$ and B are separated.

(3) Find the components of the followings:
(i) $[0, 1] \cup [2, 3]$ with usual distance.
(ii) $(0, 1) \cup \{2, 3\}$ with usual distance .
(iii) $\mathbb{Q}$

(iv) $\mathbb{R} \setminus \mathbb{Q}$
(v) $[0, 1]$ with distance metric.
(vi) $\{1, 2, 3\}$ with any metric.
(vii)$\mathbb{N}$ with usual distance .
(viii) $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q}$ or $y \in \mathbb{Q}\}$ with Euclidean distance in $\mathbb{R}^2$.

(4) Find the connected subsets of the following metric spaces:
(i) (X, d) where d is discrete metric.
(ii) (X, d) where X is a finite set.
(iii)$(\mathbb{N}, d)$ where d is usual distance in $\mathbb{R}$.
(iv) $(\mathbb{Q}, d)$ where d is usual distance in $\mathbb{R}$

(5) Show that the following subsets of $(\mathbb{R}^2, d)$ (d being Euclidean distance) are not connected.
(i) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$
(ii) $\{(x, y) \in \mathbb{R}^2 : y \neq 0\}$
(iii) $\mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 : y = 6\}$

(6) Prove or disprove:

(i) If A, C are connected subsets of a metric space and $A \subseteq B \subseteq C$, then B is connected.
(ii) If A° and ∂A are connected then A is connected.
(iii) If A, B are connected then $A \cup B, A \cap B$ are connected.
(iii) An open ball in a metric space is connected.
(iv) If A is a connected subset of a metric space (X, d), then A° and ∂A (boundary of A) are connected.

Topology of Metric Spaces and Real Analysis: Practical 3.4
Path Connectedness, Convex sets, Continuity and Connectedness
Objective Questions 3.4

. (Revised Syllabus 2018-19)

(1) Let (X, d) be a connected metric space. If $f : X \longrightarrow \mathbb{R}$ (d usual) is a non-constant continuous function .Then, $f(X)$ is
(a) finite set (b) countable set. (c) singleton set (d) uncountable set.

(2) The unit circle $S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ is (distance Euclidean)

(a) Compact and Connected
(b) Compact but not Connected
(c) Connected but not Compact
(d) neither Compact nor Connected

(3) Let (X, d) be a finite metric space ,$|X| \geq 2$. If $f : X \longrightarrow \mathbb{R}$ (usual distance) is a continuous function, then
(a) $|f(X)| \geq 2$
(b) $f(X)$ is connected.
(c) If $f(X)$ is connected then f is a constant function.
(d) None of these.

(4) Let A be a non-empty connected subset of $\mathbb{R}^2$ (distance Euclidean). Let $S = \{\|a\| : a \in A\}$. If every element in S is a rational number then
(a) A is a singleton set.
(b) Each point in A lies on a circle C_r where $C_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$ for some $r \in \mathbb{Q}$
(c) Each point in A lies on a parabola $x^2 = ry$ for some $r > 0$.
(d) None of the above.

(5) Let (X, d) be a connected metric space and $A \subseteq X$ consider $\chi_A : X \longrightarrow \mathbb{R}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

(a) If χ_A is continuous on X, then A is a finite set.
(b) If χ_A is continuous on X, then $A = \emptyset$ or $A = X$.
(c) If χ_A is continuous on X then A is a non-empty proper subset of X.
(d) None of the above.

(6) If $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function, then $f([a, b])$ is
(a) $(0, M]$ for some $M > 0$ (b) (m, M) for some $m, M \in \mathbb{R}$
(c) $[m, M]$ for $m, M \in \mathbb{R}$ (d) None of these.

(7) Which of the following statements is false in $\mathbb{R}^n$?
(a) Continuous image of a compact set is compact.
(b) Continuous image of a connected set is connected.
(c) Continuous image of a path connected set is path connected.
(d) None of the above.

(8) Let (X, d) be a connected metric space which is not bounded. Let $x_0 \in X$ and $A_r = \{x \in X : d(x, x_0) = r\}(r > 0)$. Then
(a) $A_r = \emptyset$ except for finitely many positive real number r.
(b) $A_r \neq \emptyset \ \forall\ r > 0$
(c) $A_r = \emptyset \ \forall\ r > 0$
(d) None of these.

(9) Let (X, d) be a connected metric space and $f : X \longrightarrow \mathbb{Z}$ be a continuous map. Then
(a) f is onto . (b) f is one-one.
(c) f is bijective. (d) f is constant

(10) $\mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}$ is not path connected if
(a) $n = 3$, (b) $n = 4$ (c) $n = 1$ (d) None of these.

(11) In $(\mathbb{R}^2, d)$ (d Euclidean distance), the following set is not path connected.

(a) $\mathbb{R}^2 \setminus \mathbb{Q} \times \mathbb{Q}$
(b) $\mathbb{R}^2 \setminus \{(0, 0)\}$
(c) $\mathbb{R}^2 - \{(x, y) : y = 0\}$
(d) $B((0, 0), r) \setminus \{(0, 0)\}$

(12) In $(\mathbb{R}^2, d)$ (d Euclidean distance), the following set is path connected.
(a) $B((0, 0), 1) \cup \{(x, y) \in \mathbb{R}^2 : y = 1\}$
(b) $B((0, 0), 1) \cup \{(x, y) \in \mathbb{R}^2 : y = 2\}$
(c) $B((0, 0), 1) \cup \{(x, y) \in \mathbb{R}^2 : x = 2\}$
(d) None of the above.

(13) Which of the following statements is false:
(a) A path connected subset of $\mathbb{R}^n$ (distance being Euclidean) is connected.
(b) A connected subset of $\mathbb{R}^n$ (distance being Euclidean) is path connected.
(c) Union of two path connected subsets A, B in $\mathbb{R}^n$ distance being Euclidean such that $A \cap B \neq \emptyset$ is again path connected.
(d) If A, B are two path connected subsets of $\mathbb{R}^n$ (distance being Euclidean) such that $A \cap B \neq \emptyset$ then $A \cap B$ is path connected.

(14) Let (X, d) and (Y, d') be metric spaces. If $f : (X, d) \longrightarrow (Y, d')$ is a continuous function, then
(a) Number of components of $(X, d) \leq$ Number of components of (Y, d').
(b) Number of components of $(X, d) \geq$ Number of components of (Y, d').
(c) Number of components of $(X, d) =$ Number of components of (Y, d').
(d) Cannot say.

(15) Let (X, d) and (Y, d') be metric spaces and $f : (X, d) \longrightarrow (Y, d')$ be a bijective continuous function, then
   (a) Number of components of $(X, d) \leq$ Number of components of (Y, d').
   (b) Number of components of $(X, d) \geq$ Number of components of (Y, d').
   (c) Number of components of $(X, d) =$ Number of components of (Y, d').
   (d) Cannot say.

Topology of Metric Spaces and real Analysis: Practical 3.4
Path Connectedness, Convex sets, Continuity and Connectedness
Descriptive Questions 3.4

(1) Prove that the following subsets of $\mathbb{R}^n$ (distance being Euclidean) are convex and hence path connected. (i) an open ball (ii) a closed ball (iii) a line

(2) Let (X, d) be a metric space and A be a proper non-empty subset of X. If the characteristic function χ_A is continuous on X, show that X is not connected.

(3) Show that $B_r((0, 0)) \setminus \{(0, 0)\}$ is path connected in $\mathbb{R}^2$ with Euclidean distance.

(4) Show that $\mathbb{R}^2 \setminus S \times S$ where S is any countable subset of $\mathbb{R}$ is path connected. (Hint: For any $x, y \in \mathbb{R}^2 \setminus S \times S$ there are uncountable lines passing through x and y).

(5) Prove or disprove :

   (a) If A is a path connected subset of $\mathbb{R}^n$ (distance being Euclidean) thenA° is path connected.
   (b) If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of path connected subsets of $\mathbb{R}^2$ (distance being Euclidean) such that $A_{n+1} \subseteq A_n \ \ \forall n \in \mathbb{N}$ and $\cap_{n\in\mathbb{N}} A_n \neq \emptyset$ then $\cap_{n\in\mathbb{N}} A_n$ is connected.

(6) (a) If (X, d) is a connected metric space and $f : X \longrightarrow \mathbb{Z}$ a continuous function, prove that f is constant.
   (b) If (X, d) is a connected metric space and (Y, d') is any metric space, Y being a finite set, then show that any continuous function $f : X \longrightarrow Y$ is constant.
   (c) Let (X, d) be a connected metric space and (Y, d_1) be a discrete metric space. Show that any continuous function $f : X \longrightarrow Y$ is constant.

(7) Let (X, d) be a connected metric space which is not bounded. Prove that for each $x_0 \in X$ and each $r > 0$, the set $\{x \in X : d(x, x_0) = r\}$ is non-empty.

(8) Give an example of a subset of $\mathbb{R}^n$ (distance being Euclidean) which is connected but not path connected.

(9) Show that if (X, d) is a connected metric space then either X is countable or X is singleton.

(10) Show that the following sets are path connected subsets of $\mathbb{R}^2$.

(i) $E = \{(x, y) \in \mathbb{R}^2, x > 0, x^2 - y^2 = 1\}$

(ii) $E_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$

(iii) $E = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$

(iv) $E = \{(x, y) \in \mathbb{R}^2 : y^2 = x\} \cup \{(x, y) \in \mathbb{R}^2 : y^2 = -x\}$

(v) $E = \{(x, y) \in \mathbb{R}^2 : y = 0\}$

(vi) $E = \{(x, y) \in \mathbb{R}^2 : 1 < 2x + y < 3\}$

(vii) $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

(viii) $E = \{(x, y) \in \mathbb{R}^2 : 0 < x < 2, 1 < y < 5\}$

Topology of Metric Spaces and Real Analysis: Practical 3.5
Pointwise and Uniform Convergence of Sequences of Functions and Properties
Objective questions 3.5

. (Revised Syllabus 2018-19)

(1) $\chi_n : \mathbb{R} \longrightarrow \mathbb{R}$ $\chi_n(x) = \begin{cases} 1 & \text{if } x \in [-n, n] \\ 0 & \text{if } x \notin [-n, n] \end{cases}$
(a) $\{\chi_n\}$ converges pointwise to 0 on $\mathbb{R}$.
(b) $\{\chi_n\}$ does not converge uniformly on $\mathbb{R}$.
(c) $\{\chi_n\}$ converges uniformly to 1 on $\mathbb{R}$.
(d) None of the above.

(2) Let $f_n(x) = \sin nx$ for $x \in \mathbb{R}$. and $g_n(x) = \dfrac{f_n(x)}{n}$ $\forall x \in \mathbb{R}$. Then
(a) $\{f_n\}$ and $\{_n\}$ are uniformly convergent on $\mathbb{R}$.
(b) $\{f_n\}$ and $\{_n\}$ are not pointwise convergent on $\mathbb{R}$.
(c) $\{g_n\}$ is uniformly convergent on $\mathbb{R}$ but $\{f_n\}$ is not.
(d) $\{f_n\}$ is uniformly convergent on $\mathbb{R}$ but $\{g_n\}$ is not.

(3) Let $f_n : [0, 1] \longrightarrow [0, 1]$ be defined by $f_n(x) = x * \chi_n(x)$ where $\chi_n(x) = \begin{cases} 0 & \text{if } x \notin \left[0, \dfrac{1}{n}\right] \\ 1 & \text{if } x \in \left[0, \dfrac{1}{n}\right] \end{cases}$
(a) $\{f_n\}$ converges uniformly to 0 on $[0, 1]$.
(b) $\{f_n\}$ converges pointwise to 1 on $[0, 1]$ but does not converge uniformly.
(c) $\{f_n\}$ converges uniformly to 1 on $[0, 1]$.
(d) None of the above.

(4) The least integer value of k for which $\left\{\dfrac{e^{-nx}}{n^k}\right\}$ is uniformly convergent on $[0, \infty)$ is
(a) 0 (b) 1 (c) -1 (d) 2

(5) If $\{f_n\}$ and $\{g_n\}$ are sequences of functions on $S, S \subseteq \mathbb{R}$ converging uniformly to f and g respectively on S then the following sequence of functions may not converge uniformly of S to the given function.
(a) $\{f_n + g_n\}$ to $f + g$. (b) $\{f_n - g_n\}$ to $f - g$. (c) $\{\lambda f_n\}$ to λf. (d) $\{f_n * g_n\}$ to $f * g$.

(6) Let $f_n(x) = \dfrac{x^n}{1 + x^n}, 0 \le x \le 1$.
(a) $\{f_n\}$ converges uniformly on $[0, 1]$
(b) $\{f_n\}$ converges uniformly on $\left[\dfrac{1}{2}, 1\right]$

(c) $\{f_n\}$ converges uniformly on $\left[0, \dfrac{1}{2}\right]$
(d) $\{f_n\}$ converges uniformly on $(0, 1]$

(7) Let $f_n(x) = \dfrac{x^n}{n} \quad \forall\, x \in [0, 1]$. Then
(a) $\{f_n\}$ converges uniformly to 0 but f_n' does not converge uniformly on $[0, 1]$.
(b) $\{f_n\}$ converges uniformly to 0 and f_n' converges uniformly to 1 on $[0, 1]$.
(c) $\{f_n\}$ does not converges uniformly on $[0, 1]$ but f_n' converges uniformly on $[0, 1]$.
(d) None of the above.

(8) Let $f_n(x) = \dfrac{x}{x+n}$ for $x \in [0, \infty)$. Show that $\{f_n\}$ does not converge uniformly on $[0, \infty)$ but converges uniformly on $[0, a]$ where $a > 0$. Also show that $\{f_n\}$ does not converge uniformly on $[a, \infty], a > 0$
(a) $\{f_n\}$ converges uniformly on $[0, \infty)$
(b) $\{f_n\}$ converges uniformly on $[a, \infty), a > 0$
(c) $\{f_n\}$ converges uniformly on $[0, a], a > 0$
(d) None of the above.

(9) $g_n(x) = x^{n-1}(1-x), 0 \leq x \leq 1$.
(a) $\{g_n\}$ is uniformly convergent on $[0, 1]$.
(b) $\{g_n\}$ is not uniformly convergent on $[0, 1]$.
(c) $\{g_n\}$ is not pointwise convergent on $[0, 1]$.
(d) None of the above.

(10) $\{f_n\}$ and $\{g_n\}, g_n \neq 0$ are real valued functions on a non-empty subset $S, S \subseteq \mathbb{R}$ which are uniformly convergent to the functions f and g respectively on S.
(a) $\{fn * g_n\}$ need not be uniformly convergent on S.
(b) $\{f_n/g_n\}$ is uniformly convergent on S.
(c) $\{f_n * g_n\}$ is uniformly convergent to $f * g$ on S if each f_n is bounded on S.
(d) $\{f_n * g_n\}$ converges uniformly to $f * g$ on S if and only if either $f \equiv 0$ or $g \equiv 0$ on S.

(11) Let $\{f_n\}$ be a sequence of real valued functions on a set S converging uniformly to a function f. Then the following statement is **not** true.
(a) Each f_n is bounded on $S \Longrightarrow f$ is bounded on S.
(b) Each f_n is differential on $S \Longrightarrow f$ is differentiable on S
(c) Each f_n is continuous on $S \Longrightarrow f$ is continuous on S.
(d) Each f_n is integrable on $S \Longrightarrow f$ is integrable on S.

(12) Let $\{f_n\}$ be a sequence of real valued $R-$integrable functions on $[a, b]$ and f be the pointwise limit of $\{f_n\}$
(a) If $\lim\limits_{n \longrightarrow \infty} \displaystyle\int_a^b f_n \neq \int_a^b f$ then $\{f_n\}$ doesn't converge uniformly to f.
(b) If $\{f_n\}$ doesn't converge uniformly to f, then $\lim\limits_{n \longrightarrow \infty} \displaystyle\int_a^b f_n \neq \int_a^b f$.

(c) If $\lim_{n\longrightarrow\infty} \int_a^b f_n \neq \int_a^b f$ then the convergence is uniform.
(d) None of the above.

(13) Let $\{f_n\}$ be a sequence of differentiable functions on (a,b). Let $\lim_{n\longrightarrow\infty} f_n(x) = f(x)$, $\lim_{n\longrightarrow\infty} f_n'(x) = g(x)$ (pointwise limits)
(a) If f is differntable on (a,b), then $f' = g$ on (a,b)
(b) If $\{f_n'\}$ converges uniformly to g, then f is differentable on (a,b) and $f' = g$.
(c) If $f' = g$ on (a,b) then $\{f_n\}$ converges uniformly to f on (a,b).
(d) If $\{f_n\}$ converges uniformly to f, then f is differentiable and $f' = g$ on (a,b)

(14) Let $f_n(x) = \begin{cases} x & \text{if } x \leq n \\ n & \text{if } x \geq n \end{cases}$
(a) $\{f_n\}$ converges uniformly on $\mathbb{R}$ to a bounded function.
(b) $\{f_n\}$ converges uniformly on $\mathbb{R}$ to an unbounded functions.
(c) $\{f_n\}$ is not pointwise convergent on $\mathbb{R}$.
(d) $\{f_n\}$ converges pointwise on $\mathbb{R}$.

(15) Let $f_n(x) = \dfrac{x}{1+nx^2}$
(a) $\{f_n\}$ converges uniformly on $\mathbb{R}$ but $\{f_n'\}$ does not converge uniformly on $\mathbb{R}$.
(b) $\{f_n\}$ converges uniformly on $\mathbb{R}$ and $\{f_n'\}$ also converges uniformly on $\mathbb{R}$.
(c) $\{f_n\}$ does not converge uniformly on $\mathbb{R}$ but $\{f_n'\}$ converges uniformly on $\mathbb{R}$.
(d) Neither $\{f_n\}$ nor $\{f_n'\}$ converge uniformly on $\mathbb{R}$.

(16) Let $f_n(x) = \dfrac{x^n}{1+x^n}$ on $[0,2]$ and $f(x) = \lim_{n\longrightarrow\infty} f_n(x)$
(a) $\{f_n\}$ converges uniformly to f on $[0,2]$ and f is continuous at $x = 1$.
(b) $\{f_n\}$ does not converge uniformly to f on $[0,2]$ and f is not continuous on $[0,1]$.
(c) $\{f_n\}$ does not converge uniformly to f on $[0,2]$ but f is continuous on $[0,2]$.
(d) None of the above.

(17) $f_n(x) = x^n$ for $x \in [0,1]$
(a) The pointwise limit of $\{f_n\}$ is not continuous on $[0,1]$
(b) $\{f_n\}$ converges pointwise on $[0,1]$ to a continuous function.
(c) $\{f_n\}$ converges uniformly on $[0,1]$ to a continuous function.
(d) None of the above.

(18) $f_n(x) = \dfrac{x^n}{n}$ for $x \in [0,1]$. Let $\lim_{n\longrightarrow\infty} f_n(x) = f(x)$, $\lim_{n\longrightarrow\infty} f_n'(x) = g(x)$
(a) $\{f_n\}$ and $\{f_n'\}$ converge uniformly on $[0,1]$.
(b) $\{f_n'\}$ converges uniformly to g on $[0,1]$
(c) $\{f_n'\}$ does not converge uniformly to g on $[0,1]$.
(d) None of the above

Topology of Metric Spaces and Real Analysis: Practical 3.5
Pointwise and Uniform Convergence of Sequences of Functions and Properties
DESCRIPTIVE QUESTIONS 3.5

(1) Show that each of the following sequences of functions converges pointwise on $(0,1)$. Identify the subintervals on which the convergence is uniform.

(i) $\dfrac{n}{nx+1}$ (ii) $\dfrac{x}{nx+1}$ (iii) $\dfrac{1}{nx+1}$

(2) Examine the following sequences of functions for pointwise and uniform convergence on $[0,1]$

(i) nxe^{-nx^2} (ii) $n^{\frac{1}{2}}x(1-x^2)^n$ (iii) $nx(1-x^2)^{n^2}$

(3) Examine the following sequences of functions for pointwise and the uniform convergence on $[0,\infty)$. In case of the convergence not being uniform, examine whether the convergence is uniform on $[0,a]$ or $[a,\infty)$ where $a>0$.

(i) e^{-nx} (ii) $\dfrac{\sin nx}{1+nx}$ (iii) x^2e^{-nx} (iv) $\dfrac{xe^{\frac{-x}{n}}}{n}$ (v) $n^2x^2e^{-nx}$

(4) $f_n:(0,\infty)\longrightarrow\mathbb{R}, f_n(x)=\dfrac{n}{1+nx}$. Then

(i) Show that $\{f_n\}$ is bounded on $(0,\infty)$ for each $n\in\mathbb{N}$.
(ii) Find the pointwise limit f of $\{f_n\}$ and show that f is not bounded on $(0,\infty)$.
(iii) Is $\{f_n\}$ uniformly convergent on $(0,\infty)$? State clearly the theorem you used.
(iv) Show that there does not exist $\alpha\in\mathbb{R}^+$ such that $|f_n(x)|\le\alpha$ for all $n\in\mathbb{N}$ and for all $x\in(0,\infty)$.

(5) Show that the following sequences of functions do not converge uniformly on the given domain.

(i) $f_n:[0,\infty)\longrightarrow\mathbb{R}, f_n(x)=\begin{cases} x & \text{if } x\le n\\ n & \text{if } x>n\end{cases}$

(ii) $f_n:[0,\infty), f_n(x)=\dfrac{nx}{1+nx^2}$.

(iii) $f_n:(0,1]\longrightarrow\mathbb{R}, f_n(x)=\begin{cases} 0 & \text{if } 0<x\le\frac{1}{n}\\ \dfrac{1}{x} & \text{if } \frac{1}{n}<x\le 1\end{cases}$

(6) $f_n : [0,1] \longrightarrow \mathbb{R}, f_n(x) = nxe^{-nx}$. Show that each f_n continuous on $[0,1]$, the pointwise limit of $\{f_n\}$ continuous on $[0,1]$ but $\{f_n\}$ does not converge uniformly convergent on $[0,1]$.

(7) Show that the following sequence of functions do not converge uniformly on the given domain.

$f_n : [0,1] \longrightarrow \mathbb{R}, f_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}$.

(8) Let $f_n : [0,1] \longrightarrow \mathbb{R}, f_n(x) = \dfrac{1}{nx+1}$.
Show that $\{f_n\}$ converges pointwise to f on $[a,b]$ and each f_n and f are $R-$integrable on $[0,1]$ with $\lim\limits_{n\longrightarrow\infty} \displaystyle\int_a^b f_n(x)\ dx = \int_a^b f(x)dx$ but $\{f_n\}$ does converges uniformly on $[0,1]$.

(9) Let $f_n : [0,1] \longrightarrow \mathbb{R}, f_n(x) = \begin{cases} n^2 & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$. Show that $\{f_n\}$ does not converge uniformly on $[0,1]$. (Hint: show that if each f_n is R-integrable on $[0,1]$ and $f_n \longrightarrow f$ pointwise on $[0,1]$ but $\lim\limits_{n\longrightarrow\infty} \displaystyle\int_a^b f_n(x)$ is not convergent.)

(10) Let $f_n : [0,1] \longrightarrow \mathbb{R}$ is defined for $n \geq 2$, $f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x \leq \frac{1}{n} \\ -n^2\left(x - \dfrac{2}{n}\right) & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \leq x \leq 1 \end{cases}$. Show that $\{f_n\}$ does not converge uniformly on $[0,1]$.
(Hint: Show that each f_n is R-integrable on $[0,1]$ and $f_n \longrightarrow f$ pointwise on $[0,1]$, but $\lim\limits_{n\longrightarrow\infty} \displaystyle\int_0^1 f_n(x) \neq \int_0^1 f(x)dx$.)

(11) Let $f_n : [-1,1] \longrightarrow \mathbb{R}, f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$. Given that $f_n \longrightarrow f$ uniformly on $[-1,1]$ where $f(x) = |x|$ for $x \in [-1,1]$. Find $\lim\limits_{n\longrightarrow\infty} \displaystyle\int_{-1}^1 f_n(x)\ dx$.

(12) Let $f_n : [0,1] \longrightarrow \mathbb{R}, f_n(x) = x + n$. Does $\{f_n\}$ converge pointwise at any $x \in [-1.1]$. Does sequence $\{f_n\}$ converge uniformly on $[-1,1]$? Show that $\{f_n'\}$ converges uniformly on $[-1,1]$.

(13) $f_n : \mathbb{R} \longrightarrow \mathbb{R}, f_n(x) = \dfrac{e^{-n^2x^2}}{n}$. Find the pointwise limit function f of $\{f_n\}$ and g of $\{f_n'\}$. Does $f_n' \longrightarrow g$ uniformly on $\mathbb{R}$?. Is $f'(0) = g(0)$?

Topology of Metric Spaces and Real Analysis: Practical 3.6
Objective Questions 3.6

. (Revised Syllabus 2018-19)

(1) The series $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ is
(a) uniformly convergent on $[0, A]$ where $A > 0$ but not on $[0, \infty)$.
(b) not uniformly convergent on $[0, A]$ where $A > 0$.
(c) uniformly convergent on $[0, \infty)$.
(d) none of the above.

(2) The series $\sum_{n=1}^{\infty} \frac{x^n}{n+1}$ is
(a) uniformly convergent on $\mathbb{R}$.
(b) not uniformly convergent on $[-a, a]$ where $0 < a < 1$
(c) uniformly convergent on $[-a, a]$ where $0 < a < 1$.
(d) none of the above.

(3) The series $\sum_{n=1}^{\infty} \frac{x^n}{x^n+1}$ is
(a) pointwise convergent on $[1, \infty)$.
(b) uniformly convergent on $[0, a], a < 1$.
(c) uniformly convergent on $[0, \infty)$.
(d) none of the above.

(4) The series $\sum_{n=1}^{\infty} \frac{x}{[(n-1)x+1)][nx+1]}$ is
(a) uniformly convergent on $[0, \infty)$.
(b) uniformly convergent on $[0, 1]$.
(c) uniformly convergent on $[a, b], a > 0$.
(d) none of the above.

(5) The series $\sum_{n=1}^{\infty}(-x)^n(1-x)$ is
(a) uniformly convergent on $\mathbb{R}$.
(b) uniformly on $[0, 1]$.
(c) uniformly convergent on $[0, a]$ where $0 \le a < 1$ but not on $[0, 1]$.
(d) none of the above.

(6) The least value of integer k for which $\sum_{n=1}^{\infty} \frac{\sin nx}{n^k}$ converges uniformly on $\mathbb{R}$ is
(a) 1. (b) 2. (c) -1. (d) none of the above.

(7) $\sum_{n=1}^{\infty} |a_n|$ is convergent then $\sum_{n=1}^{\infty} a_n x^n$ is

(a) uniformly convergent on $\mathbb{R}$.
(b) uniformly convergent on any bounded interval.
(c) uniformly convergent on $[-a, a]$, where $0 \leq a < 1$.
(d) none of the above.

(8) The series $\sum_{n=1}^{\infty} x^n(1-x)$

(a) converges uniformly to x on $[0, a]$, where $0 \leq a < 1$.
(b) converges uniformly on $[0, 1)$.
(c) is not pointwise convergent at $x = 1$.
(d) none of the above.

(9) The series $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ (a) converges uniformly on $(0, \infty)$.

(b) converges uniformly on $[a, \infty), a > 0$.
(c) does not converge uniformly on $[a, \infty), a > 0$.
(d) none of the above.

(10) The series $\sum_{n=1}^{\infty} \frac{1}{(nx)^2}$

(a) converges uniformly on $\mathbb{R} \setminus \{0\}$.
(b) does not converge uniformly on $[a, \infty), a > 0$.
(c) converges uniformly on $[a, \infty), a > 0$.
(d) none of the above.

(11) Consider the series $\sum_{n=1}^{\infty} x^n(1-2x^n)$. Then

(a) $\int_0^1 \sum_{n=1}^{\infty} x^n(1-2x^n)dx \neq \sum_{n=1}^{\infty} \int_0^1 x^n(1-2x^n)dx$.
(b) $\int_0^1 \sum_{n=1}^{\infty} x^n(1-2x^n)dx = \sum_{n=1}^{\infty} \int_0^1 x^n(1-2x^n)dx$.
(c) it converges uniformly on $[0, 1]$ and can be integrated term by term.
(d) none of the above.

(12) Let $f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$. Then

(a) None of the below statements are true.
(b) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is not uniformly convergent on $[0, 1]$ and cannot be integrated term by term.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is uniformly convergent on $[0,1]$ and can be integrated term by term.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is not uniformly convergent on $[0,\delta]$, where $0 \leq \delta < 1$ and

$$\lim_{\delta \longrightarrow 1} \lim_{n \longrightarrow \infty} \int_0^{\delta} \sum_{k=1}^{n} \frac{\cos kx}{k^2} dx \neq \lim_{\delta \longrightarrow 1} \int_0^{\delta} \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} dx$$

(13) The power series expansion for $\displaystyle\int_0^x e^{-t^2} dt$ is

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(n+1)!}$ (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!(n+1)}$

(14) If R is the radius of convergence of power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$, then radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} c_n^k x^{nk}$ is (a) R^k (b) R (c) $R^{\frac{1}{k}}$ (d) $\dfrac{1}{R^k}$

(15) If R is the radius of convergence of power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$, then radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} c_n x^{nk}$ is (a) R^k (b) R (c) $R^{\frac{1}{k}}$ (d) $\dfrac{1}{R^k}$

(16) If R is the radius of convergence of power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ then the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} c_n x^n$ is (a) R^2 (b) R (c) 0 (d) ∞

(17) $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R_1 and $\displaystyle\sum_{n=0}^{\infty} b_n x^n$, has radius of convergence R_2.

Let $C_n = \begin{cases} a_n & \text{if } n \text{ is even} \\ b_n & \text{if } n \text{ is odd} \end{cases}$. Then the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ is

(a) $R_1 + R_2$ (b) $\min\{R_1, R_2\}$ (c) $\max\{R_1, R_2\}$ (d) None of the above.

(18) Let R be the radius of convergence of power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$, then the following powe series does not have radius of convergence $\mathbb{R}$.

(a) $\displaystyle\sum_{n=0}^{\infty} (-1)^n c_n x^n$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{c_n}{n} x^n$ (c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n c_n^2 x^n$ (d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n n c_n x^n$

(19) Let $\sum_{n=0}^{\infty} c_n x^n$ be a power series with integer coefficients such that $c_n \neq 0$ for infinitely many n. If R is the radius of convergence of $\sum_{n=0}^{\infty} x^n$, then

(a) $R = 0$ (b) $R = \infty$ (c) $R \leq 1$ (d) $R \geq 1$

(20) Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $|x| < R$. If $f(x)$ is an even function for $|x| < R$, then

(a) $c_n = 0 \ \forall\ n \in \mathbb{N}$.
(b) $c_n = 0$ when n is even.
(c) $c_n = 0$ when n is odd.
(d) None of the above.

(21) If $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 1, then

(a) the power series converges at $x = 1$ and $x = -1$.
(b) the power series diverges at $x = 1$ and $x = -1$.
(c) the power series converges at $x = 1$ and diverges at $x = -1$.
(d) none of the above.

(22) Consider the power series $\sum_{n=0}^{\infty} c_n x^n$, for which $c_n = \begin{cases} \frac{1}{2^k} & \text{if } n = 2k \\ 3^{k+1} & \text{if } n = 2k+1 \end{cases}$.

Then the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$ is

(a) 2 (b) $\sqrt{2}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

(23) If α is a non-zero real number then the radius of convergence of $\alpha^n x^n$ is

(a) $|\alpha|$ (b) $\frac{1}{|\alpha|}$ (c) 0 (d) ∞

(24) If α and β are real numbers such that $0 < |\beta| < |\alpha|$ then radius of convergence of $\sum_{n=0}^{\infty} (\alpha^n + \beta^n) x^n$ is (a) $|\alpha|$ (b) $\frac{1}{|\alpha|}$ (c) $|\beta|$ (d) $\frac{1}{|\beta|}$

(25) The series expansion $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ is valid if

(a) $|x| \leq 1$ (b) $|x| \leq A$ for $A > 0$ (c) $|x| < 1$ (d) $x > 0$

(26) The series expansion $1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = \frac{1}{(1-x)^2}$ is valid in

(a) $\mathbb{R}$ (b) $(-1, 1)$ (c) $[-1, 1)$ (d) $[a, b]$ for any $a, b \in \mathbb{R}, a < b$

(27) Let $E(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} + \cdots$ for $x \in \mathbb{R}$. Then $\lim\limits_{x \longrightarrow \infty} x^n E(-x) =$
(a) 1 (b) 0 (c) ∞ (d) -1

(28) Let $E(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}, C(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, x \in \mathbb{R}$. Then

(a) $E(x), C(x), S(x)$ are one-one
(b) Only $E(x)$ is one-one
(c) $C(x), S(x)$ are one-one
(d) None of the above.

(29) Let $L : (0, \infty) \longrightarrow \mathbb{R}$ be defined as $L(E(x)) = x$ and $E(L(y)) = y, x \in \mathbb{R}$.

(a) $L(1-y) = -\displaystyle\sum_{n=1}^{\infty} \frac{y^n}{n}$
(b) $L(y) = \displaystyle\sum_{n=1}^{\infty} \sum_{n=0}^{\infty} n y^n$
(c) $L(y) = \displaystyle\sum_{n=1}^{\infty} \frac{y^n}{n+1}$
(d) None of the above.

(30) Let $L : (0, \infty) \longrightarrow \mathbb{R}$ be defined as $L(E(x)) = x$ and $E(L(y)) = y, x \in \mathbb{R}$. Then
(a) L is represented by power series on $(0, 1)$. (b) L is represented by power series on $(0, \infty)$. (c) L is not represented by power series. (d) None of the above.

(31) Let $\cosh x = \dfrac{E(x) + E(-x)}{2}$ and $\sinh x \dfrac{E(x) - E(-x)}{2}, x \in \mathbb{R}$. Then the following identity is not true.
(a) $\sinh(-x) = -\sinh x, \cosh(-x) = -\cosh x$ (b) $\sinh(x + y) = \sinh x \cosh(y) + \cosh x \sinh y$ (c) $\dfrac{d}{dx} \sinh x = \cosh x$ (d) $\cosh^x + \sinh^2 x = 1$

Topology of Metric Spaces and Real Analysis: Practical 3.6
Descriptive Questions 3.6

(1) Show that $\displaystyle\sum_{n=1}^{\infty} x^n (1-x)$ converges uniformly to x on $[0, a]$, where $0 \leq a < 1$, but $\displaystyle\sum_{n=1}^{\infty} x^n (1-x)$ is not uniformly convergent on $[0, 1)$.

(2) Show that the series $\displaystyle\sum_{n=1}^{\infty} (-x)^n (1-x)$ converges uniformly on $[0, 1]$.

(3) (i) Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(nx)^2}$ does not converge uniformly on $\mathbb{R} \setminus \{0\}$ but converges uniformly on $[a, \infty), a > 0$.

(ii) Show that $\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$ is uniformly convergent on $\mathbb{R}$.

(iii) Show that $\sum_{n=1}^{\infty} \frac{1}{x^n+1}$ is uniformly convergent on $[a, \infty), a > 1$.

(iv) Show that $\sum_{n=1}^{\infty} \frac{x^n}{x^n+1}$ is uniformly convergent on $[0, a], a < 1$ but not pointwise convergent on $[1, \infty)$.

(v) Show that $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ does not converge uniformly on $(0, \infty)$ but converges uniformly on $[a, \infty), a > 0$.

(4) Show that each of the following series of functions converges uniformly on the indicated interval.

(i) $\sum_{n=1}^{\infty} e^{-nx}x^n, [0, A], A > 0$.

(ii) $\sum_{n=1}^{\infty} \frac{e^{-nx}}{n}, x \in [a, \infty), a > 0$.

(iii) $\sum_{n=1}^{\infty} e^{-nx}$ on $[a, \infty), a > 0$.

(5) If $\sum_{n=1}^{\infty} |a_n| < \infty$, then the series $\sum_{n=1}^{\infty} a_n \cos nx$ and $\sum_{n=1}^{\infty} a_n \sin nx$ converge on $\mathbb{R}$.

(6) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n(x^2+n)}{n^2}$ converges uniformly every bounded subset of $\mathbb{R}$.

(7) Show that $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}, p \leq 1$ is uniformly convergent on $S = [-\pi, -a] \cup [a, \pi], a > 0$.

(8) (i) $\sum_{n=1}^{\infty} 2x \left[\frac{e^{\frac{-x^2}{n^2}}}{n^2} - \frac{e^{\frac{-x^2}{(n+1)^2}}}{(n+1)^2} \right]$ in $[a, b]$. Show that the series converges uniformly to $2xe^{-x^2}$ on $[a, b]$. Hence show that $\sum_{n=1}^{\infty} \int_a^b 2x \left[\frac{e^{\frac{-x^2}{n^2}}}{n^2} - \frac{e^{\frac{-x^2}{(n+1)^2}}}{(n+1)^2} \right] dx = e^{-a} - e^{-b}$.

(ii) $\sum_{n=1}^{\infty} x^n(1-2x^n)$. Show that the series does not converge pointwise at $x=1$ but converges pointwise to $\frac{x}{1+x}$ on $[0,1)$. Show that $\int_0^1 \frac{x}{1+x}dx \neq \sum_{n=1}^{\infty}\int_0^1 x^n(1-2x^n)dx$. Hence show that the series does not converge uniformly on $[0,1)$. State the result you used . (Let D be a bounded subset of $\mathbb{R}$ and let $f : D \longrightarrow \mathbb{R}$ be a function. We say that f is integrable over D if f is a bounded function and if there are $a, b \in \mathbb{R}$ with $D \subseteq [a, b]$ such that the function $f^* : [a, b] \longrightarrow \mathbb{R}$ defined by

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in D \\ 0 & \text{otherwise} \end{cases}$$

is integrable on $[a, b]$. In this case, the Riemann integral of f over D is defined by $\int_D f(x)dx = \int_a^b f^*(x)dx$.

Reference: A Course in Calculus an Real Ananlysis, Sudhir R. Ghorpade, Balmohan V. Limaye,Second Edition, Springer, pg. no. 216)

(iii) $\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]$ in $[0,1]$.

Show that $\int_0^1 \left[\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]\right] dx = \sum_{n=1}^{\infty}\int_0^1 \left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right] dx.$

but $\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]$ does not converge uniformly on $[0,1]$.

(9) Show that $\sum_{n=1}^{\infty}\frac{1}{n^3+n+x^2}$ is uniformly convergent on $\mathbb{R}$ and check that it can be differentiated term by term.

(10) Find the radius of convergence of each of the following power series.

(i) $\sum_{n=0}^{\infty} n^3x^n$ (iii) $\sum_{n=0}^{\infty}\frac{n^3}{3^n}x^n$ (v) $\sum_{n=0}^{\infty}\frac{e^n}{n+1}x^n$

(ii) $\sum_{n=0}^{\infty}\frac{2^n}{n!}x^n$ (iv) $\sum_{n=0}^{\infty}(n^3-5n^2+7n-2)x^n$ (vi) $\sum_{n=0}^{\infty}\frac{x^n}{(n+1)^{\sqrt{n}}}$

(11) Find the interval of convergence of the following power series.

(i) $\sum_{n=0}^{\infty}\frac{(x-1)^{n-1}}{3^n n^2}$ (ii) $\sum_{n=0}^{\infty}\frac{n!(x-2)^n}{n^n}$ (iii) $\sum_{n=0}^{\infty}\frac{(x^2-1)^n}{2^n}$

(iv) $\displaystyle\sum_{n=0}^{\infty} \frac{(3x+6)^n}{n!}$ (v) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^{n-1}}{n}$

(12) Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$, where $c_n = \dfrac{h(h-1)\cdots(h-n+1)}{n!}$.

(13) Consider the power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ with integer coefficients. If $c_n \neq 0$ for infinitely many n, then show that its radius of convergence is at most 1.

(14) Give an example of a power series with radius of convergence $= 5$ and interval of convergence $= (3, 13)$.

(15) If $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ is a power series such that $0 < \alpha < |c_n| < \beta \quad \forall\, n \in \mathbb{N}$ where $\alpha, \beta \in \mathbb{R}$, find its radius of convergence.

(16) Let $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ and $\displaystyle\sum_{n=0}^{\infty} b_n x^n$ be power series such that

$$a_n = \begin{cases} 1 & \text{if } n \text{ is square of an integer} \\ 0 & \text{otherwise} \end{cases} \qquad b_n = \begin{cases} 1 & \text{if } n = k! \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ and $\displaystyle\sum_{n=1}^{\infty} b_n x^n$.

(17) If $0 < |\alpha| < |\beta|$ then find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} (\alpha^n + \beta^n) x^n$ and $\displaystyle\sum_{n=0}^{\infty} (\alpha^n + \beta^n) x^n$.

(18) Show that $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

(19) By differentiating a suitable power series term by term, obtain the formula,

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = \frac{1}{(1-x)^2}$$

for $-a \leq x \leq a$. What should be the value of $'a'$ so that term by term differentiation is valid?

(20) If $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ for $x \in \mathbb{R}$ and $\frac{d}{dx}(\sin x) = \cos x, \forall x \in \mathbb{R}$, then show that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

(21) Show by integrating the series for $\frac{1}{1+x}$, that $\log(1+x) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{n}$.

(22) By integrating a suitable powe series over an interal $[0,t]$, where $0 \le t \le 1$, show that $\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{n!(n+2)}$.

(23) For $|x| < 1$, show that $\sin^{-1} x = \sum_{n=0}^{\infty} \frac{1.3.5.\cdots.(2n-1)x^{2n+1}}{2.4.\cdots.(2n)(2n+1)}$.

(24) For $|x| < 1$, show that $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$.

(25) Find a series expansion for $\int_0^x e^{-t^2} dt$ for $x \in \mathbb{R}$.

(26) If $\sum_{n=0}^{\infty} |a_n| < \infty$, prove that $\int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}$.

Topology of Metric Spaces and Real Analysis: Practical 3.7
Miscellaneous.

. Revised Syllabus 2018-19

UNIT I : Continuous functions on Metric Spaces

(1) Let (X,d) and (Y,d') be metric spaces. Show that $f : X \to Y$ is continuous at $p \in X$ if and only if for each sequence (x_n) in X converging to p, the sequence $(f(x_n))$ converges to $f(p)$ in Y.

(2) Let (X,d) and (Y,d') be metric spaces and $f : X \longrightarrow Y$. Show that the following statements are equivalent.

(i) f is continuous on X.

(ii) For each open subset G of $Y, f^{-1}(G)$ is an open subset of X.

(iii) For each closed subset F of $Y, f^{-1}(F)$ is a closed subset of X.

(3) Let (X,d) and (Y,d') be metric spaces. Show that $f : X \longrightarrow Y$ is continuous at $p \in X$ if and only if for each sequence (x_n) in X converging to p, the sequence $(f(x_n))$ converges to $f(p)$ in Y.

(4) Let (X,d) and (Y,d') be metric spaces. Show that f is continuous at $x \in X$ if and only if for each open subset U of Y containing $f(x), \exists$ an open subset V of X containing x such that $f(V) \subseteq U$.

(5) Let (X,d) and $(Y,d'), (Z,d'')$ be metric spaces. If $f : X \longrightarrow Y$ is continuous and $g : Y \longrightarrow Z$ is continuous, then show using $\epsilon - \delta$ definition or sequential criterion that $g \circ f : X \longrightarrow Z$ is continuous. Give an example to show that converse of the above statement is not true.

(6) Let (X,d) and (Y,d') be metric spaces. Show that $f : X \longrightarrow Y$ is continuous on X if and only if for each subset A of $X, f(\overline{A}) \subseteq \overline{(f(A))}$.

(7) Let (X,d) and (Y,d') be metric spaces. Show that $f : X$ßY is continuous on X if and only if for each subset B of $Y, \overline{(f^{-1}(B))} \subseteq f^{-1}(\overline{B})$.

(8) Let (X,d) and (Y,d') be metric space and $f : X \longrightarrow R$ (usual distance) be a continuous function. If $f(x_0) > 0$ for some $x_0 \in X$, show that $f(x) > 0, \forall x \in B(x_0, \delta)$.

(9) Let (X,d) and (Y,d') be metric spaces. When is $f : X \longrightarrow Y$ said to be uniformly continuous? Give an example to show that a continuous map need not be uniformly continuous.

(10) Let (X,d) and (Y,d') be metric spaces. If $f, g : X \longrightarrow Y$ are continuous functions, then show that $F = \{x \in X : f(x) = g(x)\}$ is a closed subset of X. Hence, deduce that if $f(x) = g(x), \forall x \in D$, where D is a dense subset of X, then $f = g$.

(11) Let (X, d) and (Y, d') be metric spaces. Show that $f : (X, d) \longrightarrow (Y, d')$ is a continuous function if and only if $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$ for each subset B of Y.

(12) Let (X, d) be a metric space and $A \subseteq X$. Using $\epsilon - \delta$ definition show that $f_A(x) = d(x, A)$ is a continuous map from (X, d) to $(\mathbb{R}, d)$ where d is the usual distance on $\mathbb{R}$.

(13) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function (distance is Euclidean) and F be a closed subset of $\mathbb{R}$. Show that $A = \{x \in F : f(x) = 0\}$ is a closed set in $\mathbb{R}$. Is the result true if F is not closed?

(14) Let (X, d) be a metric space. Show that $f : (X, d) \longrightarrow (\mathbb{R}, d)$ (where d is usual distance) is continuous if and only if $f^{-1}(-\infty, a)$ and $f^{-1}(a, \infty)$ are both open in (X, d) for each $a \in \mathbb{R}$.

(15) Show that the metrics d and d_1 on a set X are equivalent if and only if $i : (X, d) \longrightarrow (X, d_1)$ and $i : (X, d_1) \longrightarrow (X, d)$ are continuous functions, where i denotes the identity map on X.

(16) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ (with respect to usual distance) and $A = \{(x, y) : y < f(x)\}, B = \{(x, y) : y > f(x)\}$. Show that f is continuous on $\mathbb{R}$ if and only if A, B are open subsets of $(\mathbb{R}^2, d)$ where d is the Euclidean distance.

(17) Let X be a finite set and d be any metric on X. Show that any function $f : X \longrightarrow Y$ is continuous, where (Y, d') is a metric space.

(18) Let (X, d) be a discrete metric space and (Y, d') be any metric space. Show that any function $f : X \longrightarrow Y$ is continuous.

(19) Show that any function $f : (\mathbb{N}, d) \longrightarrow (X, d')$ is continuous, where d is usual distance on $\mathbb{N}$ and (X, d') is any metric space.

(20) Show that any function $f : (\mathbb{Z}, d) \longrightarrow (X, d')$ is continuous, where d is usual distance on $\mathbb{Z}$ and (X, d') is any metric space.

(21) (X, d) and (Y, d') are metric space and $f : X \longrightarrow Y$ is continuous. Give examples to show that

 (i) G is an open subset of X does not imply $f(G)$ is an open subset of Y.
 (ii) F is a closed subset of X does not imply $f(F)$ is a closed subset of Y.
 (iii) (x_n) is a Cauchy sequence in X does not imply the sequence $(f(x_n))$ is a Cauchy in Y.

(22) Let (X, d) be a metric space and (Y, d') be any metric spaces. If $f : (X, d) \longrightarrow (Y, d')$ is a continuous function, then show that $f(X)$ is a compact set.

(23) Let (X, d) and (Y, d') be metric spaces and $f : X \longrightarrow Y$ be continuous. If (X, d) is a compact metric space, then show that $f : X \longrightarrow Y$ is uniformly continuous.

(24) Let (X, d) be a complete metric space. $T : X \longrightarrow X$ be a contraction map. Then show that T has a fixed point.

(25) Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a mapping such that $T^m = T \circ T \circ T \circ \ldots \circ T(m\text{times})$ is a contraction for some fixed m then show T has an unique fixed point.

(26) Let (X, d) be a compact metric space and $T : X \longrightarrow X$ be such that $d(T(x), T(y)) < d(x, y)$ then show that T has unique fixed point in X.

UNIT II : Connected sets

(1) Let (X, d) be a metric space. Prove that the following statements are equivalent:

- (i) X can be expressed as a union of two non-empty separated sets.
- (ii) X can be expressed as a union of two non-empty disjoint closed sets.
- (iii) X can be expressed as a union of two non-empty disjoint open sets.
- (iv) There is a non-empty proper subset of X which is both open and closed.

(2) Show that A is a connected subset of $\mathbb{R}$ with respect to the usual distance if and only if it is an interval.

(3) Let (X, d) be a connected metric space and (Y, d') be any metric space. If $f : (X, d) \longrightarrow (Y, d')$ is a continuous function, then show $f(X)$ is a connected set.

(4) Show that a metric space (X, d) is connected if and only if every continuous function $f : X \longrightarrow \{1, -1\}$ is constant.

(5) If a metric space (X, d) is connected and A is a non-empty proper subset of X, then show that δA, boundary of A is non-empty.

(6) Show that a metric space (X, d) is connected if and only if for each $a, b \in X$, there is a connected subset E of X such that $a, b \in E$.

(7) Let (X, d) be a metric space. If A is a connected subset of X, and $A \subseteq B \subseteq \overline{A}$ then show that B is connected. Hence, show that $\overline{A}$ is connected. Give an example to show that if A, C are connected subset of X and $A \subseteq B \subseteq C$ then B need not be connected.

(8) If A and B are connected subset of a metric space (X, d), and $A \cap B \neq \emptyset$, then show that $A \cup B$ is connected. Give an example to show that $A \cap B$ need not be connected.

(9) Let (X, d) be a metric space. If $\{A_\alpha : \alpha \in \Lambda\}$ is a family of connected subsets of X such that $\cap_{\alpha \in \Lambda} A_\alpha \neq \emptyset$, then show that $\cup_{\alpha \in \Lambda} A_\alpha$ is connected.

(10) Let (X, d) be a metric space. If $\{A_n : n \in \mathbb{N}\}$ is a family of connected subsets of X such that $A_n \cap A_{n+1} \neq \emptyset$ for each $n \in \mathbb{N}$, then show that $\cup_{n \in \mathbb{N}} A_n$ is connected.

(11) Prove that an open ball in $\mathbb{R}^n$ is a convex set. (The distance being Euclidean). Hence, deduce that it is path connected.

(12) Show that a path connected subset of $\mathbb{R}^n$ is connected.

(13) Let A and B be path connected subsets of a metric space (X, d) such that $A \cap B \neq \emptyset$. Show that $A \cup B$ is path connected.

(14) Let (X, d) and (Y, d') be metric spaces. If (X, d) is path connected and $f : X \longrightarrow Y$ is continuous, show that $f(X)$ is path connected.

(15) Let (X, d) be a metric space and A be a non-empty subset of X.
Prove or disprove: IfA is connected, then A°, and ∂A are connected. Give an example to show that A° and ∂A may be connected, but A may not be connected.

(16) Let (X, d) be a metric space. If A is connected subset of X, then show that $\overline{A}$ is connected. Give an example to show that A° may not be connected.

UNIT III : Sequences and series of functions

(1) **Mn Test**: A sequence $\{f_n\}$ of real valued functions on S ($S \subseteq \mathbb{R}$) converges uniformly to a function $f : S \longrightarrow \mathbb{R}$ on S if and only if, $\lim\limits_{n \longrightarrow \infty} M_n = 0$ where
$M_n = \sup\{|f_n(x) - f(x)| : x \in S\}$. Hence show that if there is a sequence (t_n) in $\mathbb{R}$ such that $|f_n(x) - f(x)| \leq t_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$ and for all $x \in S$ such that $t_n \longrightarrow 0$, then $f_n \longrightarrow f$ uniformly on S.

(2) State and prove Cauchy Criterion for uniform convergence of sequences of functions.

(3) Let $\{f_n\}$ be a sequence of real valued functions defined on a set $S \subseteq \mathbb{R}$ such that $f_n \longrightarrow f$ uniformly on S. If each f_n is bounded on S, then prove the following.

 (i) f is bounded on S.
 (ii) there exists $\alpha \in \mathbb{R}^+$ such that $|f_n(x)| \leq \alpha$ for all $n \in \mathbb{N}$ and for all $x \in S$.
 (iii) $\sup\{f_n(x) : x \in S\} \longrightarrow \sup\{f(x) : x \in S\}$.
 (iv) $\inf\{f_n(x) : x \in S\} \longrightarrow \inf\{f(x) : x \in S\}$.

(4) Let $\{f_n\}$ be a sequence of real valued continuous functions defined on a subset S of $\mathbb{R}$ such that $f_n \longrightarrow f$ uniformly on S. Then prove the following.

 (i) f is continuous on S.
 (ii) For any $p \in S$, $\lim\limits_{n \longrightarrow \infty} \lim\limits_{x \longrightarrow p} f_n(x) = \lim\limits_{x \longrightarrow p} \lim\limits_{n \longrightarrow \infty} f_n(x)$.

(5) Let $\{f_n\}$ be a sequence of real valued $R-$integrable functions defined on $[a, b]$ such that $f_n \longrightarrow f$ uniformly on [a, b] . Then prove that f is $R-$ integrable on $[a, b]$ and $\lim\limits_{n\longrightarrow\infty} \int_a^b f_n(t)\, dt = \int_a^b \lim\limits_{n\longrightarrow\infty} f_n(t)\, dt.$

(6) $\{f_n\}$ is a sequence of real valued $R-$integrable functions on $[a, b]$ converging uniformly to f on $[a, b]$. If $F_n(x) = \int_a^x f_n(t)\, dt$ then prove that $\{F_n\}$ converges uniformly to F on $[a, b]$ where $F(x) = \int_a^x f(t)dt.$

(7) Let $\{f_n\}$ and $\{g_n\}$ be sequences of real valued bounded functions on S subset of $\mathbb{R}$. If $\{f_n\}$ and $\{g_n\}$ converge uniformly to f and g respectively on S, then prove that $\{f_n * g_n\}$ is uniformly convergent on S.

(8) Let $\{f_n\}$ be a sequence of real valued continuously differentiable functions on $[a, b], a < b$ such that $\{f_n(x_0)\}$ is convergent for some $x_0 \in [a, b]$ and $\{f_n'\}$ converges uniformly on $[a, b]$. Then

   (i) there is a continuously differentiable function f on $[a, b]$ such that $f_n \longrightarrow f$ uniformly on $[a, b]$ and
   (ii) $f_n' \longrightarrow f'$ uniformly on $[a, b]$.

(9) Let $\{f_n\}$ be a sequence of differentiable real valued functions on a bounded interval I. If $\{f_n(x_0)\}$ is convergent for some $x_0 \in I$ and $\{f_n'\}$ converges uniformly to g on I then $\{f_n\}$ converges uniformly on I and if $\{f_n\}$ converges uniformly to f on I then f is differentiable on I and $f' = g$ on I.

(10) State and prove Cauchy Criterion for Uniform Convergence of a Series $\sum\limits_{n=0}^{\infty} f_n$ of real valued functions on a subset S of $\mathbb{R}$.

(11) State and prove Weierstrass M-Test for the convergence of a series $\sum\limits_{n=1}^{\infty} f_n$ of real valued functions defined on subset S of $\mathbb{R}$.

(12) Let $\{f_n\}$ be a sequence of real-valued bounded functions on a set $S \subseteq \mathbb{R}$. If the series $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly to the sum function f on S then prove that f is also bounded on S.

(13) If $\{f_n\}$ is a sequence if real valued continuous functions on $S, S \subseteq \mathbb{R}$ such that $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly to f on S, then prove that f is continuous on S, and for $p \in S$, $\sum\limits_{n=1}^{\infty} \lim\limits_{x \longrightarrow p} f_n(x) = \lim\limits_{x \longrightarrow p} \sum\limits_{n=1}^{\infty} f_n(x)$.

(14) Let $\sum\limits_{n=1}^{\infty} f_n$ be a series of $R-$integrable functions on $[a, b]$, converging uniformly to f on $[a, b]$, then prove that f is $R-$integrable on $[a, b]$ and $\int_a^b f(x)\ dx = \sum\limits_{n=1}^{\infty} \int_a^b f_n(x)\ dx$.

(15) If $\{f_n\}$ is a sequence of differentiable functions on $[a, b]$ such that each f_n' is continuous on $[a, b]$ and if $\sum\limits_{n=1}^{\infty} f_n$ converges to f pointwise on $[a, b]$ and $\sum\limits_{n=1}^{\infty} f_n'$ converges uniformly on $[a, b]$ then prove that $f'(x) = \sum\limits_{n=1}^{\infty} f_n'(x) \quad$ for $a \leq x \leq b$.
NOTE: For Q. No. (12) to (15), the corresponding result about uniform convergence of sequence of functions can be used directly.

(16) If the power series $\sum\limits_{n=0}^{\infty} c_n x^n$ converges at $x_1 \in \mathbb{R}, x_1 \neq 0$ and diverges at $x_2 \in \mathbb{R}$ then the power series $\sum\limits_{n=0}^{\infty} |c_n x^n|$ converges for all $x \in \mathbb{R}$ with $|x| < |x_1|$ and diverges for all $x \in \mathbb{R}$ with $|x| > |x_2|$.

(17) A power series $\sum\limits_{n=1}^{\infty} c_n x^n$ is either absolutely convergent for all $x \in \mathbb{R}$, or there is a unique real number $r \geq 0$ such that the series is absolutely convergent for each $x \in \mathbb{R}$ with $|x| < r$ and is divergent for each $x \in \mathbb{R}$ with $|x| > r$.

(18) Let $\sum\limits_{n=0}^{\infty} c_n x^n$ be a power series with coefficients in $\mathbb{R}$. Let $\alpha = \limsup\limits_{n \longrightarrow \infty} \Big| c_n \Big|^{\frac{1}{n}}$. Then the radius of convergence r of $\sum\limits_{n=0}^{\infty} c_n x^n$ is $\dfrac{1}{\alpha}$ (if $\alpha = 0, r = \infty$ and if $\alpha = \infty, r = 0$) (Statement Only).

Definition: **limit superior of a sequence** $\left(\limsup_{n\longrightarrow\infty} a_n\right)$: Let (a_n) be a sequence in $\mathbb{R}$.

i. If (a_n) is not bounded above then $\limsup_{n\longrightarrow\infty} a_n = \infty$

ii. If (a_n) is bounded above then for each $n \in \mathbb{N}$, define, $M_n = \sup\{a_k : k \geq n\}$. Then sequence (M_n) is monotonic decreasing. If sequence (M_n) is bounded below then it is convergent. In such case, $\limsup_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} M_n$.

iii. If sequence (M_n) is not bounded below then $\limsup_{n\longrightarrow\infty} a_n = -\infty$.

It can be proved that if sequence (a_n) is convergent then $\limsup_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} a_n$

(19) Let $\sum_{n=0}^{\infty} c_n x^n$ be a power series with coefficients in $\mathbb{R}$ and there exist $n_0 \in \mathbb{N}$ such that $c_n \neq 0$, $\forall n \geq n_0$. Let $\alpha = \lim_{n\longrightarrow\infty} \left|\frac{c_{n+1}}{c_n}\right|$. Then the radius of convergence r of $\sum_{n=0}^{\infty} c_n x^n$ is $\frac{1}{\alpha}$ (if $\alpha = 0, r = \infty$ and if $\alpha = \infty, r = 0$) (Statement Only).

(20) Let r be the radius of convergence of a power series $\sum_{n=1}^{\infty} c_n x^n$. If $s \in \mathbb{R}$ is such that $0 < s < r$, then prove that the power series converges uniformly on $[-s, s]$. Further, let $f : (-r, r) \longrightarrow \mathbb{R}$ be the sum function of the power series $\sum_{n=0}^{\infty} c_n x^n$ then prove that

(i) f is continuous on $(-r, r)$.

(ii) For every $x \in (-r, r)$, $\int_0^x f(t)dt = \sum_{n=0}^{\infty} c_n \frac{x^{n+1}}{n+1}$

(iii) f is differentiable on $(-r, r)$ and $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ for $x \in (-r, r)$.

(iv) f is infinitely differentiable on $(-r, r)$, and $c_n = \frac{f^{(n)}(0)}{n!}$ for $n \in \mathbb{N}, c_0 = f(0)$.

Numerical Methods 1
Lagrange's Linear, quadratic and higher order Interpolation, Iterated interpolation, Newton's divided difference interpolation

Numerical Methods Objective Questions 1

(1) The Lagrange's quadratic interpolating polynomial obtained in two different ways

(a) are not identical at all.

(b) may be different in form but are identical otherwise.

(c) always in the same form.

(d) None of the above.

(2) The Lagrange's quadratic interpolating polynomial of $f(x)$ such that $f(0) = 1$, $f(1) = 3$, $f(3) = 55$ using Lagrange's interpolation is

(a) $8x^2 - 6x + 1$ (b) $8x^2 + 6x - 1$ (c) $-8x^2 - 6x + 1$ (d) None of these

(3) If $f(1) = 0.2675$, $f(2) = 0.5287$, then the appropriate value of $f(1.5)$ using Newton's divided difference interpolation polynomial is

(a) 0.2981 (b) 0.3981 (c) 1.3981 (d) None of these

(4) The interpolating polynomial of $f(x)$ such that $f(1) = -2$, $f(2) = 1$, $f(3) = 2$ using iterated interpolation is

(a) $x^2 + 6x - 7$ (b) $x^2 - 6x - 7$ (c) $-x^2 + 6x - 7$ (d) None of these

(5) Which of the following statement is true.

(a) Lagrange's and Newton's divided difference polynomials are two different forms of the same polynomial.

(b) Lagrange's and iterated interpolation polynomials are two different forms of the same polynomial.

(c) Newton's divided difference and iterated interpolation polynomials are two different forms of the same polynomial.

(d) All the above

(6) Let $f(-2) = 46$, $f(-1) = 4$, $f(3) = 156$. Using Lagrange's quadratic interpolation formula, the value of $f(0)$ is

(a) $-\frac{1}{15}$ (b) $-\frac{3}{15}$ (c) $\frac{1}{15}$ (d) $\frac{1}{15}$.

(7) Given $P(1) = 1$, $P(3) = 27$ and $P(4) = 64$, the value of $I_{0,1}(2)$ is

(a) 12 (b) 13 (c) 14 (d) 15.

(8) If $f(x) = \cos x$, $x_0 = 0.2$ and $x_1 = 0.3$, then $f[x_0, x_1]$ is

(a) 0.2473 (b) 0.2483 (c) 0.32483 (d) None of these.

(9) If $f(x) = \cos x$, $x_0 = 0.2$, $x_1 = 0.3$ and $x_2 = 0.4$, then $f[x_0, x_1, x_2]$ is

(a) -0.4772 (b) -0.5772 (c) -0.4882 (d) -0.7472

(10) The value $\lim_{x_1 \to x_0} f[x_0, x_1]$ is

(a) $f'(x_1)$ (b) $f'(x_0)$ (c) 0 (d) x_0

Numerical Methods Descriptive Questions 1

(1) Let $f(-2) = 46$, $f(-1) = 4$, $f(1) = 4$, $f(3) = 156$. Use Lagrange's interpolation formula to estimate the value of $f(0)$.

(2) Find the Lagrange's interpolating polynomial for the following data:

x	0	2	3	6
$f(x)$	659	705	729	804

(3) Find the Newton's divided difference interpolating polynomial for the following data:

x	-1	1	4	7
$f(x)$	-2	0	63	342

Also interpolate at $x = 5$.

(4) Find the Iterated interpolating polynomial for the following data:

x	-1	2	4	5
$f(x)$	-5	13	255	635

Also interpolate at $x = 3$.

(5) Find the unique polynomial $P(x)$ of degree 2 or less such that $P(1) = 1$, $P(3) = 27$, $P(4) = 64$, using each of the following methods:

(i) Lagrange's interpolating formula.

(ii) Newton divided difference formula, and

(iii) Iterated interpolation formula.

(6) If $f(x) = \dfrac{1}{x}$ for $x = x_1, x_2$ and x_3, find the second order divided difference $f[x_1, x_2, x_3]$.

(7) If $f(x) = \dfrac{1}{x^2}$ for $x = x_1, x_2, x_3$ and x_4, find the third order divided difference $f[x_1, x_2, x_3, x_4]$.

(8) Given $f(3) = 5$, $f(7) = 10$, find the linear interpolating polynomial using

(i) Lagrange interpolation

(ii) Iterated interpolation

(iii) Newton's divided difference interpolation

Hence find an approximation value of $f(4)$.

(9) Given $\ln(1) = 0$, $\ln(4) = 1.3863$ and $\ln(6) = 1.791$. Estimate $\ln(2)$ using linear and quadratic interpolation formula. Find the error.

(10) Given $f(x) = e^x$ and take $x_0 = 1$, $x_1 = 1.1$. Use linear interpolation formula to calculate and approximate value of $f(1.04)$ and $f(1.05)$ and hence obtain a bound on truncation error at the midpoint of x_0 and x_1.

(11) (i) For the data points $(0.82, 2.270500)$ and $(0.83, 2.293319)$. Find unique polynomial $P_1(x)$ of degree 1 or less and hence evaluate $P_1(0.826)$.

(ii) For the data points $(0, -1)$, $(1, -1)$ and $(2, 7)$ find unique polynomial $P_2(x)$ of degree 2 or less and hence evaluate $P_2(1.5)$.

(12) Find Lagrange's interpolation polynomial for the following

x	0	2	3	6
$f(x)$	659	705	729	804

(13) The function $y = f(x)$ is given at the points $(7, 3)$, $(8, 1)$, $(9, 1)$ and $(10, 9)$. Find the value of y for $x = 9.5$ using Lagrange's interpolation formula.

(14) The following values of the function $f(x)$ for values of x are given as $f(1) = 4$, $f(2) = 5$, $f(7) = 5$, $f(8) = 4$. Find the value of $f(6)$ and also the value of x for which $f(x)$ is maximum or minimum by Lagrange's interpolation formula.

(15) Find the value of $\tan 33^o$ by Lagrange's interpolation formula if $\tan 30^o = 0.5774$, $\tan 32^o = 0.6249$, $\tan 35^o = 0.7002$, $\tan 38^o = 0.7813$.

(16) The following values of the function $y = f(x)$ are given:

$$f(0) = 4, \quad f(1) = 3, \quad f(3) = 6.$$

Use Lagrange's quadratic interpolation formula to determine

(i) $f(1.5)$ (ii) $\dfrac{dy}{dx}$ at 0.5 (ii) $\displaystyle\int_0^4 y\,dx$

(17) Find the unique polynomial $P(x)$ of degree 2 or less such that $P(1) = 2$, $P(2) = 26$, $P(4) = 62$, using each of the following methods:

(i) Lagrange's quadratic interpolation formula

(ii) Newton's divided difference formula.

(iii) Iterated interpolation formula.

(18) Use Lagrange's interpolation formula to find the value of y at $x = 2.7$ from the following data:

x	2	2.5	3
$y = \log_e x$	0.69315	0.91629	1.09861

Also estimate the error in interpolating polynomial.

(19) By considering the limit of the three points Lagrange's interpolation formula relative to $x_0, x_1, x_0+\varepsilon$ as $\varepsilon \to 0$ obtain the formula

$$f(x) = \frac{(x-x_0)^2}{(x_1-x_0)^2} f(x_1) + \frac{(x-x_0)(x-x_1)}{x_0-x_1} f'(x_0) - \frac{(x-x_1)(x-2x_0+x_1)}{(x_0-x_1)^2} f(x_0)$$

(20) Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$ and $\log_{10} 661 = 2.8202$. Find by divided difference formula the value of $\log_{10} 656$.

(21) Find the polynomial of lowest possible degree which assumes the values $1245, 33, 5, 9$ and 1335 at $x = -4, -1, 0, 2$ and 5. Also find the nature of the polynomial at abscissa 1 by Divided Difference Formula.

(22) Apply Newton's divided difference formula to find the value of $f(8)$ if $f(1) = 3$, $f(3) = 31$, $f(6) = 223$, $f(10) = 1011$, $f(11) = 1343$.

(22) Given that

x	1	3	4	5
y_n	1	27	81	729

Find y_5 using Newton's divided difference formula.

(23) Use Newton's divided difference formula to find $f(7)$ if $f(3) = 24$, $f(5) = 120$, $f(8) = 504$, $f(9) = 720$, $f(12) = 1716$.

(24) Given

x	0	1	2	5
$f(x)$	2	3	12	147

What is the form of the function?

(25) Find the function u_x in power of $(x-1)$, given that $u_0 = 8$, $u_1 = 11$, $u_4 = 68$, $u_5 = 123$ using divided difference formula.

(26) Find the function u_x in power of $(x-4)$, where $u_0 = 8$, $u_1 = 11$, $u_4 = 68$, $u_5 = 125$ using divided difference formula.

(27) Find $f(x)$ as a polynomial in powers of $x - 5$ using the following data:

x	0	2	3	4	7	9
$f(x)$	5	26	58	112	466	922

Numerical Methods 2
Finite difference operators, Interpolating polynomial using finite differences

Numerical Methods Objective Questions 2

(1) The forward difference operator Δ is equal to

(a) $1-E$ (b) $1+E$ (c) $1-E^{-1}$ (d) $E-1$.

(2) If $f(1)=5$, $f(2)=7$, $f(3)=8$, then

(a) $\Delta f(2)=1$, $\nabla f(2)=2$ and $Ef(2)=8$
(b) $\Delta f(2)=2$, $\nabla f(2)=1$ and $Ef(2)=8$
(c) $\Delta f(2)=1$, $\nabla f(2)=1$ and $Ef(2)=3$
(d) None of these

(3) If $f(4)=-1$, $f(6)=3$, $f(8)=5$, then

(a) $\delta f(5)=3$, $\mu f(5)=1$ (b) $\delta f(5)=4$, $\mu f(5)=1$
(c) $\delta f(5)=5$, $\mu f(5)=2$ (d) None of these

(4) If $f(1)=0.227$, $f(3)=0.528$, $f(5)=0.729$, then

(a) $\Delta f(3)=0.301$, $\nabla f(2)=0.201$, $\delta f(2)=0.378$ and $\mu f(2)=0.301$
(b) $\Delta f(3)=0.378$, $\nabla f(2)=0.301$, $\delta f(2)=0.301$ and $\mu f(2)=0.201$
(c) $\Delta f(3)=0.201$, $\nabla f(2)=0.301$, $\delta f(2)=0.301$ and $\mu f(2)=0.378$
(d) None of these

(5) Which of the following statement is **false**:

(a) $f''(x)\approx\dfrac{\Delta^2 f(x)}{h^2}$ (b) $f''(x)\approx\dfrac{\nabla^2 f(x)}{h^2}$
(c) $f''(x)\approx\dfrac{\delta^2 f(x)}{h^2}$ (d) $f''(x)\approx\dfrac{\mu^2 f(x)}{h^2}$

(6) Which of the following statement is **True**:

(a) $\Delta f(x_0)=\nabla f(x_1)=\delta f(x_{1/2})$
(b) $\Delta f(x_0)\neq\nabla f(x_1)$, $\Delta f(x_0)=\delta f(x_{1/2})$
(c) $\Delta f(x_0)=\nabla f(x_1)$, $\Delta f(x_0)\neq\delta f(x_{1/2})$
(d) $\Delta f(x_0)\neq\nabla f(x_1)$, $\Delta f(x_0)\neq\delta f(x_{1/2})$

(7) Which of the following statement is true

(a) $\Delta=E(1-\nabla)$ (b) $\delta=E(1-\nabla)$ (c) $1=E(1-\nabla)$ (d) $\mu=E(1-\nabla)$.

(8) If $f(x)=x^3-2x+4$ and interval of differencing is unity then

(a) $\Delta^2 f(x)=0$ (b) $\Delta^3 f(x)=0$ (c) $\Delta^4 f(x)=0$ (d) None of the these

(9) The Gregory-Newton forward difference Interpolating polynomial corresponds to the data $f(0.1) = 1.4$, $f(0.2) = 1.56$ is

(a) $1.6x + 1.24$ (b) $1.6x - 1.24$
(c) $1.6x + 1.56$ (d) None of these

(10) The value of $f(0.2)$ using Gregory-Newton forward difference Interpolation for the data $f(0.1) = -1.699$, $f(0.3) = -1.073$ is

(a) -2.638 (b) -1.386 (c) 1.386 (d) None of these

(11) The value of $f(0.15)$ using Gregory-Newton backward difference Interpolation for the data $f(0) = -1.5$, $f(0.1) = -1.27$ is

(a) 1.2585 (b) -1.2585 (c) -1.3275 (d) None of these

(12) The Gregory-Newton backward difference Interpolating polynomial corresponds to the data $f(0.4) = 2$, $f(0.5) = 2.28$ is

(a) $2.8x - 0.88$ (b) $2.8x + 0.88$
(c) $2.8x + 3.68$ (d) None of these

(13) The backward difference operator ∇ is equal to

(a) $E^{-1}\Delta$ (b) $E\Delta$ (c) $E - \Delta$ (d) $E^{-1} + \Delta$

(14) Which of the following statement is true?

(a) $\delta = \Delta\nabla$ (b) $\delta^2 = \Delta + \nabla$ (c) $\delta^2 = \Delta\nabla$ (d) $\delta^2 = \dfrac{\Delta}{\nabla}$

(15) Which of the following statement is true?

(a) $\delta = E^{1/2}\Delta$ (b) $\delta = E^{-1/2}\nabla$ (c) $\delta = E\Delta$ (d) $\delta = E^{1/2}\nabla$

Numerical Methods Descriptive Questions 2

(1) Evaluate the following:

(i) $\Delta e^{2x}\log(3x)$

(ii) $\Delta\left(\dfrac{x^2}{\cos 2x}\right)$

(iii) $\Delta\left(\dfrac{e^x}{x!}\right)$

Take $h = 1$.

(2) Prove that

(i) $\Delta \log f(x) = \log\left(1 + \dfrac{\Delta f(x)}{f(x)}\right)$

(ii) $\Delta^n\left(a^{cx+d}\right) = a^{cx+d}(a^{ch} - 1)^n$ and hence evaluate $\Delta^2\left(\dfrac{a^{2x} + a^{4x}}{(a^{2h} - 1)^2}\right)$

(iii) $\Delta^n \cos(ax+b) = \left(2\sin\frac{ah}{2}\right)^n \cos\left(ax+b+n\left(\frac{ah+\pi}{2}\right)\right)$ and hence evaluate $\Delta^4 \cos^3 2x$.

(3) Show that $y_{a+nh} = y_a + \binom{n}{1}\Delta y_a + \binom{n}{2}\Delta^2 y_a + \cdots + \Delta^n y_a$ where y_a is a polynomial in x. Using above formula, find $f(x)$ if

x	0	1	2	3	4	5
$f(x)$	-1	3	19	53	111	199

Hence find the value of $f(8)$.

(4) Evaluate the following, interval of differencing being unity:

(i) $(2\Delta+3)(E+2)(3x^2+2)$

(ii) $(\nabla+\Delta)^2(x^2+x+1)$

(5) Let $f(E) = a_0E^n + a_1E^{n-1} + a_2E^{n-1} + \cdots + a_n$ and interval of differencing being unity. Show that

$$f(E)e^x = e^x + f(e).$$

(6) Given that

x	1	2	3	4	5	6
y_n	2	5	10	17	26	37

Find the value of $\nabla^3 y_6$.

(7) Estimate the missing term in the following:

x	1	2	3	4	5	6	7
y	2	4	8	–	32	64	128

(8) Prove that

(i) $y_4 = y_3 + \Delta y_2 + \Delta^2 y_1 + \Delta^3 y_1$

(ii) $y_4 = y_0 + 4\Delta y_0 + 6\Delta^2 y_{-1} + 10\Delta^3 y_{-1}$

(9) u_x is a function of x for which fifth differences are constants and $u_1 + u_7 = -786$, $u_2 + u_6 = 686$, $u_3 + u_5 = 1088$. Find u_4.

(10) Show that

(i) $\Delta^r E^k = \nabla^r E^{k+r}$ where r is an integer.

(ii) $\Delta^j f_0 = \sum_{k=0}^{j} (-1)^k \binom{j}{k} f_{j-k}$ where $j \in \mathbb{N}$.

(iii) $f_j = \sum_{k=0}^{j} \binom{j}{k} \Delta^k f_0$ where $j \in \mathbb{N}$.

(11) Show that

(i) $\mu\delta = \frac{1}{2}(\Delta+\nabla)$

(ii) $\delta^2 = \Delta - \nabla$

(iii) $\mu^2 = 1 + \frac{1}{4}\delta^2$

(iv) $\delta = E^{-1/2}\Delta$

(12) Prove that

(i) $e^x = \frac{\Delta^2}{E} e^x \frac{Ee^x}{\Delta^2 e^x}$

(ii) $\Delta^n \left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x(x+1)(x+2)\cdots(x+n)}$ for all $n \in \mathbb{N}$.

(13) If $f(x) = e^{2x}$, then show that

(i) $\Delta^3 f(x) = (e^{2h} - 1)^3 e^{2x}$

(ii) $\nabla^3 f(x) = (1 - e^{-2h})^3 e^{2h}$

(14) If $f(x_{-1}) = y_{-1}$, $f(x_0) = y_0$, $f(x_1) = y_1$ and $f(x_2) = y_2$, then show that $\delta^2 y_0 = y_1 - 2y_0 + y_{-1}$.

(15) Construct the interpolating polynomial that fits the data

x	0	0.1	0.2	0.3	0.4	0.5
$f(x)$	− 1.5	− 1.27	− 0.98	− 0.63	− 0.22	0.25

using the Gregory-Newton forward or backward interpolation. Hence or otherwise estimate the values of $f(x)$ at $x = 0.15, 0.25, 0.45$.

(16) Using the Newton's backward difference interpolation, construct the interpolating polynomial that fits the data

x	0.1	0.3	0.5	0.7	0.9	1.1
$f(x)$	− 1.699	− 1.073	− 0.375	0.443	1.429	2.631

Estimate the value of $f(x)$ at $x = 0.6$ and $x = 1.0$.

(17) The following data represents the function $f(x) = e^x$.

x	1	1.5	2.0	2.5
$f(x)$	2.7183	4.4817	7.3891	12.1825

Estimate the value of $f(2.25)$ using the (i) Newton's forward difference interpolation and (ii) Newton's backward difference interpolation. Compare with the exact value.

(18) The following data represents the function $f(x) = \cos(x+1)$.

x	0.0	0.2	0.4	0.6
$f(x)$	0.5403	0.3624	0.1700	− 0.0292

Estimate the value of $f(0.5)$ using the Newton's backward difference interpolation. Compare with the exact value.

(19) Ordinate $f(x)$ of a normal curve in terms of standard deviation x are given as

x	1.00	1.02	1.04	1.06	1.08
$f(x)$	0.2420	0.2371	0.2323	0.2275	0.2227

Find the ordinate for standard deviation $x = 1.025$.

(20) Using Newton's forward interpolation formula estimate the population for the year 1988 from the table.

year	1973	1983	1993	2003	2013
population	98752	132285	168076	195690	246050

(21) Find the value of an annuity at $5\frac{3}{8}\%$ given the following table using Newton's backward interpolation formula

Rate	4	4.5	5	5.5	6
Annuity value	172.2903	162.8389	153.7245	145.3375	137.6483

(22) The population of a town is as follows:

year	1972	1982	1992	2002	2012
population in crores	20	24	29	36	46

Estimate the increase in population during the period 1996 to 2002 using Newton's backward interpolation formula.

(23) For linear interpolation, in case of equispaced tabular data, show that the error does not exceed 1/8 of the second difference.

(24) Determine the step size that can be used in the tabulation of $f(x) = \sin(x)$ in the interval $[0, \pi/4]$ at equally spaced nodal points so that the truncation error of the quadratic interpolation is less than 5×10^{-8}.

(25) Given that $f(0) = 1, f(1) = 3, f(3) = 55$, find the unique polynomial of degree 2 or less, which fits the given data. Find the bound on the error.

(26) The following values of the function $f(x) = \sin(x) + \cos(x)$, are given

x	10^o	20^o	30^o
f(x)	1.1585	1.2817	1.3660

Construct the quadratic interpolating polynomial that fits the data. Hence find $f(\pi/12)$. Compare with the exact value.

(27) The following data are part of a table for $f(x) = (\cos x)/x$.

x	0.1	0.2	0.3	0.4
f(x)(in radians)	9.9500	4.9003	3.1845	2.3027

Calculate $f(0.12)$, (i) by interpolating directly from the table, (ii) by first tabulating $xf(x)$ and then interpolating from the table. Explain the difference between the results.

(28) The following data are part of a table for $g(x) = \sin x/x^2$:

x	0.1	0.2	0.3	0.4	0.5
g(x)	9.9833	4.9667	3.2836	2.4339	1.9177

Calculate $g(0.25)$ as accurately as possible

(a) by interpolating directly in this table.

(b) by first tabulating $xg(x)$ and then interpolating in that table,

(c) explain the difference between the results in (a) and (b) respectively.

(29) In the following problems, find the maximum value of the uniform mesh size h that can be used to tabulate $f(x)$ on $[a, b]$, using quadratic interpolation such that $|\text{Error}| \leq \varepsilon$.

(i) $f(x) = (2 + x)^4$, $[a, b] = [1, 2], \varepsilon = 10^{-4}$.

(ii) $f(x) = e^{x+1}$, $[a, b] = [0, 1], \varepsilon = 10^{-4}$.

(iii) $f(x) = x^2e^x$, $[a, b] = [0, 1], \varepsilon = 5 \times 10^{-6}$.

(iv) $f(x) = x^2$, $[a, b] = [5, 10], \varepsilon = 10^{-5}$.

(30) In the following problems, find the maximum value of the uniform mesh size h that can be used to tabulate $f(x)$ on $[a, b]$, using cubic interpolation such that $|\text{Error}| \leq \varepsilon$.

(i) $f(x) = e^x$, $[a, b] = [1, 2.5], \varepsilon = 10^{-4}$.

(ii) $f(x) = \cos 2x$, $[a, b] = [0, \pi/4], \varepsilon = 10^{-6}$.

(iii) $f(x) = xe^x$, $[a, b] = [1, 2], \varepsilon = 5 \times 10^{-5}$.

Numerical Methods 3
Piecewise linear and quadratic interpolation, Newton's bivariate interpolation for equi-spaced points

Numerical Methods Objective Questions 3

(1) The piecewise linear interpolation polynomial in the interval $[0, 1]$ for the data $f(0) = 1$, $f(1) = 2$, $\ldots$ is

(a) $1 + x$ (b) $1 - x$ (c) $1 + 2x$ (d) $1 + 3x$

(2) The value of $f(1.5)$ using piecewise linear interpolation polynomial in the interval $[1, 2]$ for the data $f(1) = 2$, $f(2) = 5$, $\ldots$ is

(a) 2.5 (b) 4.5 (c) 3.5 (d) 1.5

(3) For the data $f(-2) = -23$, $f(0) = 1$, $f(1) = 4$, $\ldots$, the piecewise quadratic interpolation polynomial in the interval $[-2, 1]$ is

(a) $-3x^2 + 6x + 1$ (b) $3x^2 - 6x + 1$
(c) $3x^2 + 6x + 1$ (d) None of these

(4) The value of $f(3)$ using piecewise quadratic interpolation polynomial in the interval $[1, 4]$ for the data $f(1) = -2$, $f(2) = -5$, $f(4) = 7$, $\ldots$ is

(a) 0 (b) -2 (c) 2 (d) None of these

(5) The problem of polynomial interpolation for functions of two independent variables is called

(a) quadratic spline interpolation.
(b) bivariate interpolation.
(c) linear interpolation.
(d) All the above.

(6) For the data $f(1, 2) = 5$, $f(1, 3) = 10$, $f(2, 2) = 8$ and $f(2, 3) = 13$, using Newton's bivariate interpolating polynomial, the value of $f(1.5, 2.5)$ is

(a) 10 (b) 9 (c) 8 (d) None of these

(7) For the data $f(2, 3) = 6$, $f(2, 4) = 11$, $f(3, 3) = 9$ and $f(3, 4) = 14$, interpolating polynomial is constructed using

(a) piecewise linear interpolation.
(b) piecewise quadratic interpolation.
(c) Newton's bivariate interpolation.
(d) All the above.

(8) For the data $f(0, 0) = 1$, $f(1, 0) = 1.414$, $f(0, 1) = 1.732$ and $f(1, 1) = 2$, using Newton's bivariate interpolating polynomial, the value of $f(0.25, 0.75)$ is

(a) 1.652 (b) 1.753 (c) 1.642 (d) None of these

(9) For the data $f(-1,1) = f(0,1) = f(-1,2) = 1$, $x_0 = -1$, $y_0 = 1$, $h = k = 1$, using Newton's bivariate interpolating polynomial, the value of $f(x,y)$ is

(a) -1 (b) 1 (c) 0 (d) None of these

(10) Which of the following statement is true

(a) $\Delta_x f(x,y) = (E_x + 1)f(x,y)$
(b) $\Delta_x f(x,y) = (1 - E_x)f(x,y)$
(c) $\Delta_x f(x,y) = (E_x - 1)f(x,y)$
(d) None of the above.

(11) Which of the following statement is true

(a) $\Delta_{yy} f(x,y) = (1 - E_y)^2 f(x,y)$
(b) $\Delta_{yx} f(x,y) = (E_y - 1)^2 f(x,y)$
(c) $\Delta_{xx} f(x,y) = (1 - E_y)^2 f(x,y)$
(d) $\Delta_{yy} f(x,y) = (1 - E_y) f(x,y)$

(12) Which of the following statement is true

(a) $\Delta_{xx} f(x,y) = (1 - E_x)^2 f(x,y)$
(b) $\Delta_{yy} f(x,y) = (1 - E_y)^2 f(x,y)$
(c) $\Delta_{xy} f(x,y) = \Delta_y \Delta_x f(x,y)$
(d) $\Delta_{xy} f(x,y) = (E_x - 1)(1 - E_y) f(x,y)$

(13) If $f(x,y) = e^{x-y}$and $h = 1$, then $\Delta_x f(x,y)$ is

(a) $e^{xy}(1 + e^y)$
(b) $e^{xy}(e^y - 1)$
(c) $-e^{xy}(1 + y)$
(d) None of the above

(14) If $f(x,y) = x^2 + y^2$ and $h = 2$, $k = 1$, then the value of $\Delta_{xy}(-1,1)$ is

(a) 2 (b) 4 (c) -4 (d) None of these

(15) If $f(x,y) = x^2 - y^2$ and $h = 1$, $k = 2$, then the value of $\Delta_{yy}(2,3)$ is

(a) 13 (b) -14 (c) 14 (d) None of these

Numerical Methods Descriptive Questions 3

(1) The following data for a function $f(x,y)$ is given.

$y \downarrow x \rightarrow$	1	2
2	5	8
3	10	13

Find $f(1.5, 2.5)$ using linear interpolation.

(2) The following data for a function $f(x,y)$ is given.

$y\downarrow x\rightarrow$	0	1	3
0	-4	-3	23
2	12	13	39

Obtain the interpolating polynomial that fits the data.

(3) Using the following data obtain the Lagrange and Newton's bivariate interpolating polynomial.

$y\downarrow x\rightarrow$	0	1	2
0	1	3	7
1	3	6	11
2	7	11	17

(4) Obtain the Newton's bivariate interpolating polynomial that fits the following data.

$y\downarrow x\rightarrow$	1	2	3
1	4	18	56
2	11	25	63
3	30	44	82

(5) Obtain the piecewise linear interpolating polynomials for $f(x)$ defined by the data:

x	1	2	4	8
$f(x)$	3	7	21	73

Hence estimate values of $f(3)$ and $f(7)$.

(6) Obtain the piecewise linear interpolating polynomials for $f(x)$ defined by the given data. Interpolate at the indicated points

(i)

x	0	1	2	3
$f(x)$	1	2	5	10

Interpolate at $x = 0.5$, 1.5 and 2.5.

(ii)

x	0.5	1.5	2.5
$f(x)$	0.125	3.375	15.625

Interpolate at $x = 1, 2$.

(7) Determine the piecewise quadratic fit $P(x)$ to $f(x) = (1+x^2)^{-1/2}$ with knots $-1, -1/2, 0, 1/2, 1$. Hence find an approximate value of $f(-0.75)$, $f(-0.52)$, $f(0.75)$ and $f(0.52)$.

(8) Determine the piecewise quadratic fit $P(x)$ to $f(x) = \cos x$ with knots $0, \pi/2, \pi, 3\pi/2, 2\pi$. Hence find an approximate value of $\cos(\pi/4)$, $\sin(3\pi/4)$, $\cos(5\pi/4)$.

(9) Obtain the piecewise quadratic interpolating polynomials for $f(x)$ defined by the given data. Interpolate at the indicated points

(i)

x	-2	0	1	3	4
$f(x)$	-23	1	4	82	193

Interpolate at $x = -0.5, 2$.

(ii)

x	-2	-1	1	2	4
$f(x)$	-29	-8	-2	-5	7

Interpolate at $x = 0.5, 3$.

(10) Obtain the Newton's bivariate interpolating polynomial that fits the following data:

$y \downarrow x \rightarrow$	4	5
4	10	14
5	15	17

Hence find $f(4.5, 4.5)$ and $f(4.12, 4.32)$.

(11) From the following data for a function $f(x, y)$ find $f(1.39, 1.99)$ and $f(1.22, 1.33)$.

$y \downarrow x \rightarrow$	1	2
1	2	2.3214
2	2.4782	3

(12) Using Lagrange's bivariate interpolation polynomial solve Exercise 4, 10, 11.

(13) Construct the Hermite interpolation polynomial that fits the data

x	$f(x)$	$f'(x)$
2	29	50
3	105	105

Interpolate $f(x)$ at $x = 2.5$.

(14) (i) Construct Hermite interpolation polynomial that fits the data

x	$f(x)$	$f'(x)$
0	0	1.0000
0.5	0.4794	0.8776
1.0	0.8415	0.5403

Estimate the value of $f(0.75)$. Find a bound on the error. If the data represents the function $f(x) = \sin x$, then find the actual error at $x = 0.75$.

(ii) Construct Hermite interpolation polynomial that fits the data

x	$f(x)$	$f'(x)$
0	4	-5
1	-6	-14
2	-22	-17

Interpolate $f(x)$ at $x = 0.5$ and $x = 1.5$.

(15) In the following problems, obtain the piecewisee cubic interpolating polynomials for the function $f(x)$ defined by the given data. Interpolate at the induced points.

(i)

x	-5	-4	-2	0	1	3	4
$f(x)$	275	-94	-334	-350	-349	-269	-94

Interpolate at $x = -3.0$ and $x = 2.0$

(ii)

x	-4	-3	-1	1	2	4	5
$f(x)$	-108	-255	-303	-303	-300	-108	225

Interpolate at $x = 0$ and $x = 3$

(iii)

x	-1	0	1
$f(x)$	25	-32	-37
$f'(x)$	-22	0	-14

Interpolate at $x = -0.5$ and $x = 0.5$

(iv)

x	0	1	2
$f(x)$	-6	-8	22
$f'(x)$	0	2	76

Interpolate at $x = 0.5$ and $x = 1.5$

(16) Obtain the piecewise cubic interpolating polynomials for the function f(x) defined by

x	-3	-2	-1	1	3	6	7
$f(x)$	369	222	171	165	207	990	1779

Hence find $f(6.5)$.

(17) Using the following values of $f(x)$ and $f'(x)$.

x	$f(x)$	$f'(x)$
-1	1	-5
0	1	-1
1	3	7

Estimate the values of $f(-0.5)$ and $f(0.5)$ using piecewise cubic Hermite interpolation.

Numerical Methods 4
Numerical Differentiation

Numerical Methods Objective Questions 4

(1) The value of $f'(1)$ using linear interpolation for the data $f(1) = 0.723$, $f(1.5) = 0.812$ is

(a) 0.718 (b) 0.178 (c) 0.187 (d) None of these.

(2) The value of $f'(0)$ using quadratic interpolation for the data $f(0) = 0.621$, $f(1.5) = 0.818$, $f(2.5) = 0.889$ is

(a) 0.6624 (b) 0.5334 (c) 0.1675 (d) None of these

(3) For the data $f(0.8) = 0.4096$, $f(1.0) = 1$, $f(1.2) = 2.0736$, the value of $f''(1)$ using quadratic interpolation is

(a) 12.08 (b) 11.92 (c) 12.00 (d) None of these.

(4) The step length h has optimal value if

(a) $|RE| = |TE|$ (b) $|RE| + |TE| = \text{minimum}$
(c) Either (a) or (b) (d) None of these

(where RE and TE denote the round off error and truncation error respectively).

(5) Which of the following statement is **false**:

(a) In numerical differentiation method, the local truncation error is proportional to some power of h.

(b) In numerical differentiation method, the round off error is inversely proportional to some power of h.

(c) In numerical differentiation method, the local truncation error is inversely proportional to some power of h.

(d) The error of approximation in the r-th order derivative at any point x using the method based on interpolation is $E^{(r)}(x) = f^{(r)}(x) - P_n^{(r)}(x)$.

(6) If $f(0.2, 0.1) = 2.0351$, $f(0.2, 0.2) = 2.0801$, $f(0.3, 0.1) = 2.0403$, $f(0.3, 0.2) = 2.1153$, then at $(0.2, 0.1)$

(a) $\dfrac{\partial f}{\partial x} = 0.052$, $\dfrac{\partial f}{\partial y} = 0.450$ (b) $\dfrac{\partial f}{\partial x} = 0.52$, $\dfrac{\partial f}{\partial y} = 0.045$
(c) $\dfrac{\partial f}{\partial x} = 0.052$, $\dfrac{\partial f}{\partial y} = 0.045$ (d) None of these.

(7) The following table gives the value of $f(x, y)$.

$y \downarrow x \rightarrow$	0.1	0.2	0.3
0.1	2.0200	2.0351	2.0403
0.2	2.0351	2.0801	2.1153
0.3	2.0403	2.1153	2.1893

Then the value of $\dfrac{\partial^2 f}{\partial x \partial y}$ at $(0.2, 0.2)$ is

(a) 1.1970 (b) 0.1197 (c) 2.9925 (d) None of these.

(8) The Jacobian matrix of the system of equations $f_1(x,y) = x^2 + xy - y^3 = 0$, $f_2(x,y) = 2x^2 + 5xy + 3y^2 = 0$ at $(1,1)$ is

(a) $\begin{pmatrix} 3 & -2 \\ 9 & 11 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 9 \\ -2 & 11 \end{pmatrix}$ (c) $\begin{pmatrix} 9 & 11 \\ 3 & -2 \end{pmatrix}$ (d) None of these.

(9) Error in linear interpolation is

(a) $E_1'(x_1) = \dfrac{x_0 - x_1}{2} f''(\xi)$, $x_0 < \xi < x_1$

(b) $E_1'(x_1) = \dfrac{x_1 - x_0}{2} f''(\xi)$, $x_0 < \xi < x_1$

(c) $E_1'(x_1) = \dfrac{x_0 - x_1}{2} f'(\xi)$, $x_0 < \xi < x_1$

(d) $E_1'(x_1) = \dfrac{x_1 - x_0}{2} f'(\xi)$, $x_0 < \xi < x_1$

(10) Given the following values of $f(x) = \ln x$:

x	2.0	2.2	2.6
$f(x)$	0.6915	0.78846	0.95551

Error on upper bound for $f'(2.0)$ using quadratic interpolation is

(a) 0.05 (b) 0.007 (c) 0.005 (d) 0.025.

(11) Given the following values of $f(x) = \ln x$:

x	2.0	2.2	2.6
$f(x)$	0.6915	0.78846	0.95551

Error on upper bound for $f''(2.0)$ using quadratic interpolation is

(a) 0.0704 (b) 0.704 (c) 0.007 (d) 0.005.

Numerical Methods Descriptive Questions 4

(1) Find $f'(1.6)$ and $f''(1.6)$ using quadratic interpolation from the following data for the function $f(x)$.

x	1.2	1.4	1.6
$f(x)$	0.5506	0.6048	0.6658

(2) Given the following values of $f(x)$, find the approximate value of $f'(6.0)$ using linear and quadratic interpolation and $f''(6.0)$ using quadratic interpolation.

x	6.0	6.1	6.4
$f(x)$	0.1750	-0.1998	-0.2596

(3) The following data for the function $f(x) = x^4$ is given by

x	0.4	0.6	0.8
$f(x)$	0.0256	0.1296	0.4096

Find $f'(0.8)$ and $f''(0.8)$ using quadratic interpolation. Compare with the exact solution. Obtain the bound on the truncation error.

(4) Consider the following the function

(i) $f(x) = x^2 + 2x$

(ii) $f(x) = e^x$.

Take $x = 2.0, 2.2, 2.6$. Find the approximate value of $f'(2.0)$ using linear and quadratic interpolation and $f''(2.0)$ using quadratic interpolation. Also obtain an upper bound on the error.

(5) The following table of values are given for a function $f(x, y)$:

$y \downarrow \ x \rightarrow$	0.1	0.2	0.3
0.1	2.0200	2.0351	2.0403
0.2	2.0351	2.0801	2.1153
0.3	2.0403	2.1153	2.1803

(i) Estimate the value of $\dfrac{\partial f}{\partial x}$ at $(0.2, 0.1)$, $\dfrac{\partial f}{\partial y}$ at $(0.2, 0.2)$ by first order and second order formulas.

(ii) Estimate the value of $\dfrac{\partial^2 f}{\partial x \partial y}$ at $(0.2, 0.2)$ using second order formula.

If the table represents the function $f(x, y) = 3\sin(xy) + \cos x + \cos y$, find the actual errors.

(6) he following table of values are given for a function $f(x, y)$:

$y \downarrow \ x \rightarrow$	0.5	0.7	0.9
0.4	2.0138	3.6693	6.6859
0.6	1.3499	2.4596	4.4817
0.8	0.9048	1.6487	3.0042

(i) Estimate the value of $\dfrac{\partial f}{\partial x}$ at $(0.5, 0.6)$, $\dfrac{\partial f}{\partial y}$ at $(0.9, 0.8)$ by first order formula.

(ii) Estimate the value of $\dfrac{\partial^2 f}{\partial x \partial y}$ at $(0.7, 0.6)$ using second order formula.

(7) The following systems of equations are given:

(a) $f_1(x, y) \equiv x^3 + xy^2 - y^3 = 0$, $f_2(x, y) \equiv xy + 5x + 6y = 0$

(b) $f_1(x, y) \equiv x^2 - y^2 - xy = 0$, $f_2(x, y) \equiv 3x^2 + 5y^2 + x = 0$

Find the Jacobian matrix of the systems of equations at $(1, 2)$ and $(0.5, 1)$.

(8) Find $y'(x), y''(x)$ using Newton forward, Newton backward, and Stirling's interpolation in the following

(i)

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
$y(x)$	7.989	8.403	8.781	9.129	9.451	9.750	10.031

At $x = 1.1$ and $x = 1.6$.

(ii)

x	0	1	2	3	4	5
$y(x)$	4	8	15	7	6	2

At $x = 0$.

(iii)

x	1	1.05	1.1	1.15	1.2	1.25	1.30
$y(x)$	1	1.0247	1.0488	1.0723	1.0954	1.1180	1.1401

At $x = 1$, $x = 1.25$, and $x = 1.15$.

(iv)

x	3	3.2	3.4	3.6	3.8	4.0
$f(x)$	-14	-10.032	-5.296	0.256	6.672	14

At $x = 3.0$.

(v)

x	0.4	0.5	0.6	0.7	0.8
$f(x)$	1.5836	1.7974	2.0442	2.3275	2.6511

At $x = 0.6$.

Numerical Analysis 5

Trapezoidal rule, Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule

Numerical Analysis Objective Questions 5

(1) If $\lambda_1, \lambda_2, \cdots \lambda_n$ are the Cote's numbers, then the value of $\lambda_1 + \lambda_2 + \cdots + \lambda_n$ is

(a) $(n-1)/h$ (b) nh (c) $nh-1$ (d) None of these.

(2) The approximate value of $\int_0^1 \frac{dx}{1+x}$ using Trapezoidal rule (based on interpolation) is

(a) 0.75 (b) 0.55 (c) 0.65 (d) None of these.

(3) The Trapezoidal rule for $\int_a^b f(x)dx$ is given by:

(a) $\frac{b-a}{2}[f(a)+f(b)]$

(b) $\frac{b+a}{2}[f(a)+f(b)]$

(c) $\frac{b+a}{2}[f(a)-f(b)]$

(d) $\frac{b+a}{2}[f(b)-f(a)]$

(4) The Simpson's $\frac{1}{3}$ rule for $\int_a^b f(x)dx$ is given by:

(a) $\frac{b-a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)]$

(b) $\frac{b+a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)]$

(c) $\frac{b-a}{6}[f(a)-4f(\frac{a+b}{2})+f(b)]$

(d) $\frac{b-a}{6}[f(a)-4f(\frac{a+b}{2})-f(b)]$

(5) The Simpson's $\frac{3}{8}$ rule for $\int_a^b ydx$ is given by:

(a) $\frac{b-a}{8}[y_{-1}+3y_0+3y_1+y_2]$

(b) $\frac{b+a}{8}[y_{-1}-3y_0+3y_1+y_2]$

(c) $\frac{b-a}{8}[y_{-1}+3y_0-3y_1+y_2]$

(d) $\frac{b-a}{8}[y_{-1}+3y_0+3y_1-y_2]$

(6) The approximate value of $\int_0^1 \frac{dx}{1+x^2}$ using Simpson's $\frac{1}{3}$ rule (based on interpolation) is

(a) 0.7533 (b) 0.6533 (c) 0.7833 (d) None of these.

(7) The highest order of polynomial integrand for which Simpsons $\frac{1}{3}$ rule of integration is exact is

(a) First (b) Second (c) Third (d) Fourth.

(8) Simpson's $\frac{3}{8}$ for integration is mainly based on the idea of

(a) Approximating $f(x)$ in $I = \int_a^b f(x)dx$ by a cubic polynomial.

(b) Approximating $f(x)$ in $I = \int_a^b f(x)dx$ by a quadratic polynomial.

(c) Converting the limit of integral limits $[a, b]$ into $[-1, 1]$.

(d) Using similar concepts as Gauss quadrature formula.

(9) Which of the following statement is true:

(a) Simpson's one-third rule can be applied when the range $[a, b]$ is divided into even number of subintervals.

(b) Simpson's three-eight rule can be applied when the range $[a, b]$ is divided into number of subintervals, which must be a multiple of3.

(c) Trapezoidal rule can be applied for any number of subintervals.

(d) All of the above.

Numerical Analysis Descriptive Questions 5

(1) Evaluate $\int_0^1 e^x dx$ using Trapezoidal, Simpson's $\frac{1}{3}$ abd $\frac{3}{8}$ rule.

(2) Calculate $\int_0^{1/2} \frac{x}{\sin x} dx$ using trapezoidal rule.

(3) Evaluate $\int_0^1 \left(1 + \frac{\sin x}{x}\right) dx$,

(i) using trapezoidal rule.

(ii) using Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule.

(4) Evaluate $\int_0^3 \frac{dx}{1+x}$ by Simpson's rule $\frac{1}{3}$ and $\frac{3}{8}$ and hence calculate $\log 2$.

(5) Evaluate $\int_0^1 xe^x dx$ by Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule.

(6) Evaluate $\int_0^1 e^x dx$ by Trapezoidal rule. Hence find the numerical value of the integral.

(7) Evaluate $\int_4^{5.2} \log_e(x) dx$ by Simpson's $\frac{3}{8}$ rule.

(8) Evaluate $\int_4^{5.4} \log_e(x) dx$ for the following data using Simpson's $\frac{3}{8}$ rule.

x	4	4.2	4.4	4.6	4.8	5	5.2
$\log_e(x)$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Numerical Analysis 6
Composite Trapezoidal and Simpson's rule

Numerical Analysis Objective Questions 6

(1) Using four intervals of equal length, the approximate value of $\int_1^4 \frac{dx}{2x+1}$ by Composite Simpson's rule is

(a) 0.75 (b) 0.55 (c) 0.65 (d) None of these.

(2) The approximate value of $\int_0^1 \sqrt{x^2+x+8}dx$ using four intervals of equal length by Composite Trapezoidal rule is

(a) 2.752 (b) 3.952 (c) 2.972 (d) None of these.

(3) The approximate value of $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}dx$ using Composite Simpson's with $h = \frac{1}{2}$ is

(a) 1.1073 (b) 1.1873 (c) 1.0673 (d) 1.0093

(4) The approximate value of $\int_0^{\pi} x \sin x dx$ using Composite Trapezoidal with five ordinates is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) π (d) 0

(5) The error term of the Trapezoidal rule is given by

(a) $-\frac{(b-a)^3}{12} f'''(\eta), \eta \in (0,1)$

(b) $-\frac{(b-a)^2}{12} f''(\eta), \eta \in (0,1)$

(c) $-\frac{(b-a)^3}{12} f''(\eta), \eta \in (0,1)$

(d) None of the above.

(6) The error of $\int_0^1 \frac{dx}{1+x}$ using Trapezoidal rule is

(a) 0.5685 (b) 0.6931 (c) 0.75 (d) None of these.

(7) The error of approximation in the Simpson's rule is given by

(a) $-\frac{(b-a)^5}{2880} f^{iv}(\eta), \eta \in (0,2)$

(b) $-\frac{(b-a)^4}{2880} f^{iv}(\eta), \eta \in (0,2)$

(c) $-\frac{(b-a)^5}{2880} f^{v}(\eta), \eta \in (0,2)$

(d) None of the above.

Numerical Analysis Descriptive Questions 6

(1) Obtain the approximate value of the following integrals by using Composite Simpson's rule and Composite Trapezoidal rule(take 7 equi-spaced ordinates):

(i) $\displaystyle\int_1^4 \frac{dx}{2x+1}$ (ii) $\displaystyle\int_0^1 \sqrt{x^2+x+8}dx$

(ii) Calculate $\displaystyle\int_0^{1/2} \frac{x}{\sin x}dx$ using Composite Trapezoidal rule with $h = 1/4$ and $h = 1/8$.

(2) The length of the curve represented by a function $y = f(x)$ on an interval $[a, b]$ is given by the integral $\displaystyle\int_a^b \sqrt{1 + [f'(x)]^2}dx$. Use the Composite Trapezoidal rule and Composite Simpson's rule with $n = 4$ to compute the length of the following curves:

(i) $f(x) = \sin(\pi x), 0 \le x \le 1$

(ii) $f(x) = e^x, 0 \le x \le 1$

(ii) $f(x) = e^{x^2}, 0 \le x \le 1$

(3) By computing the integral $\displaystyle\int_0^1 \frac{dx}{1+x^2}$ by Composite Simpson's rule and hence compute the value of π correct to six decimal places.

(4) Evaluate $\displaystyle\int_0^6 \frac{dx}{1+x^2}$ by Composite Simpson's rule.

(5) Evaluate $\displaystyle\int_0^1 \frac{dx}{1+x^2}$ by Composite Simpson's rule with $h = \frac{1}{6}$

(6) Using six intervals of equal length, obtain the approximate value of $\displaystyle\int_0^1 \frac{dx}{1+x}$ by using Composite Trapezoidal rule and Composite Simpson's rule. Hence obtain the approximate value of $\log_e 2$.

Numerical Analysis Practical- Semester VI

Miscellaneous Theoretical Questions

Unit 1

(1) (a) Show that the Langrange's quadratic interpolating polynomial $P(x)$ for the function $f(x)$ with interpolating conditions $f(x_i) = P(x_i), 0 \leq i \leq 2$ is given by

$$P(x) = \ell_0(x)f(x_0) + \ell_1(x)f(x_1) + \ell_2(x)f(x_2)$$

where $\ell_i(x), 0 \leq i \leq 2$ are the Langrange fundamental polynomial.

(b) If $P(x) = \sum_{i=0}^{n} \ell_i(x)f(x_i)$ is the Langrange's interpolating polynomial of degree n, then show that

(i) $P(x) = \sum_{i=0}^{n} \frac{\pi(x)}{(x - x_i)\pi'(x)} f(x_i)$.

(ii) $\sum_{i=0}^{n} \ell_i(x) = 1$

(c) With usual notation, show that the Newton's Divided Difference Interpolating polynomial $P_n(x)$ for the function $f(x)$ with nodal points $x_0, x_1, \ldots, x_n$ is given by

$$P_n(x) = f[x_0] + (x - x_0)f[x_0, x_1] + \cdots + (x - x_0)\cdots(x - x_{n-1})f[x_0, x_1, \cdots x_n]$$

(2) (a) State Rolle's theorem

(b) If f is a function s.t.

(i) f is $(n + 1)$ times continuously differentiable on $[a, b]$.

(ii) $x_0, x_1, \cdots, x_n$ are $(n + 1)$ distinct points in $[a, b]$

(iii) $P_n(x)$ is an interpolating polynomial of degree at most n that interpolates $f(x)$ in $[a, b]$.

Then show that for every $x \in [a, b]$ there is $\xi = \xi(x)$ in (a, b) where $x \neq x_i$, for $i = 0, \cdots, n$ s.t.

$$E_n(f; x) = f(x) - P_n(x) = \frac{f^{n+1}(\xi(x))}{(n + 1)!} \prod_{i=0}^{n} (x - x_i)$$

(c) Determine the maximum truncation error for linear interpolation.

(d) Derive the error formula when tabulated values are equally spaced.

(e) Determine the step size h that can be used in the tabulation of a function $f(x)$, $a \leq x \leq b$, at equally spaced nodal points so that the truncation error of the quadratic interpolation is less than ε.

(f) Determine the step size h that can be used in the tabulation of a function $f(x)$, $a \leq x \leq b$, at equally spaced nodal points so that the truncation error of the cubic interpolation is less than ε.

(3) (a) Explain the two operators Δ and E used in numerical analysis and obtain the relation between the two.

(b) Prove the following relations.

(i) $\Delta - \nabla = \Delta\nabla$.

(ii) $\Delta + \nabla = \Delta/\nabla - \nabla/\Delta$.

(iii) $\sum_{k=0}^{n-1} \Delta^2 f(x_k) = \Delta f(x_n) - \Delta f(x_0)$.

(iv) $\Delta(f(x_i)g(x_i)) = f(x_i)\Delta(g(x_i)) + g(x_{i+1})\Delta f(x_i)$.

(v) $\Delta(f(x_i)/g(x_i)) = \dfrac{g(x_i)\Delta(f(x_i)) - f(x_i)\Delta g(x_i)}{g(x_i)g(x_{i+1})}$.

(vi) $\Delta(1/f(x_i)) = -\dfrac{\Delta f(x_i)}{f(x_i)f(x_{i+1})}$

(c) If $f(x) = e^{ax}$, then show that

(i) $\Delta^n f(x) = (e^{ah} - 1)^n e^{ax}$

(ii) $\nabla^n f(x) = (1 - e^{-ah})^n e^{ax}$

(d) Prove the following relations.

(i) $\mu^2 = 1 + \frac{1}{4}\delta^2$

(ii) $\delta\mu = \frac{1}{2}(\Delta + \nabla)$.

(iii) $\Delta = \dfrac{1}{2}\delta^2 + \delta\sqrt{1 + \dfrac{1}{4\delta^2}}$.

(iv) $\Delta((f(x-1)\Delta g(x-1)) = \Delta(f(x)\nabla g(x))$.

(v) $\Delta\nabla f(x) = \nabla\Delta f(x) = \delta^2 f(x)$.

(vi) $\delta = \Delta E^{-\frac{1}{2}} = \nabla E^{\frac{1}{2}}$.

(4) (a) Show that $E = 1 + \Delta$ and deduce the Gregory-Newton forward difference interpolating polynomial with usual notation

$$P_n(x) = \sum_{i=0}^{n} {}^{u}C_i \Delta^i f(x_0)$$

(b) Show that $E = 1 - \nabla$ and deduce the Gregory-Newton backward difference interpolating polynomial with usual notation

$$P_n(x) = \sum_{i=0}^{n} (-1)^i \ {}^{-u}C_i \nabla^i f(x_0)$$

(c) Derive Stirling's central difference formula for interpolation and discuss its important uses.

Unit 2

(1) (a) Derive Piecewise linear interpolation formula.

(b) Derive Piecewise quadratic interpolation formula.

(c) Derive Piecewise cubic. interpolation formula.

(2) Derive Lagrange's bivariate interpolating polynomial for a function $f(x, y)$ defined at $(m+1)(n+1)$ distinct points $(x_i, y_j), i = 0, 1, \cdots, m, j = 0, 1, \cdots n$

(3) Show that $E = 1 + \Delta$ and deduce the Newton's bivariate interpolating polynomial $P(x, y)$ for equispaced points for the function with usual notation is given by

$$P(x, y) = f(x_0, y_0) + \left[\frac{1}{h}(x - x_0)\Delta_x + \frac{1}{k}(y - y_0)\Delta_y\right] f(x_0, y_0) + \cdots$$

(4) (a) Using Langrange's interpolating polynomial $P_n(x) = \sum_{k=0}^{n} \ell_k(x) f(x_k)$ where $\ell_k(x)$ is the Langrange's fundamental polynomial, show that $P_1'(x) = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ using linear interpolation.

(b) Using Langrange's interpolating polynomial $P_n(x) = \sum_{k=0}^{n} \ell_k(x) f(x_k)$ where $\ell_k(x)$ is the Langrange's fundamental polynomial, show that

$$P_2'(x_0) = \frac{2x_0 - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

using quadratic interpolation.

(c) Using Langrange's interpolating polynomial

$$P_n(x) = \sum_{k=0}^{n} \ell_k(x) f(x_k)$$

where $\ell_k(x)$ is the Langrange's fundamental polynomial, show that

$$P_2''(x) = 2\left[\frac{f(x_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}\right]$$

using quadratic interpolation and hence find error approximation at x_0.

(5) Given $(x_0, y_0), \cdots (x_n, y_n)$, derive first and second derivative of a function $f(x)$, using

(i) Newton's forward Interpolation formula.

(ii) Newton's backward interpolation formula.

(iii) Stirling's Interpolation formula.

(6) Given $z = f(x, y)$, if the value of $f(x, y)$ is known at $(x_i, y_i)_{i=0,\cdots,n}$ then

(i) Define $\left(\dfrac{\partial f}{\partial x}\right)_{(x_i, y_i)}$ and $\left(\dfrac{\partial f}{\partial y}\right)_{(x_i, y_i)}$ using first order and second order formula.

(ii) Define $\left(\dfrac{\partial^2 f}{\partial x \partial y}\right)_{(x_i, y_i)}$ and $\left(\dfrac{\partial^2 f}{\partial y \partial x}\right)_{(x_i, y_i)}$ using first order and second order formula.

Unit 3

(1) (a) Derive the Newton-Cotes Quadrature formula $\int_a^b f(x)dx = \sum_{k=0}^{n} \lambda_k f(x_k)$, where

$$\lambda_k = \frac{(-1)^{n-k}}{k!(n-k)!} h \int_0^n s(s-1)\cdots(s-k+1)(s-k-1)\cdots(s-n)ds.$$

(b) Derive the Newton-Cotes Quadrature formula $\int_a^b f(x)dx = \sum_{k=0}^{n} \lambda_k f(x_k)$ and deduce the trapezoidal rule $\int_a^b f(x)dx = \frac{b-a}{2}[f(a)+f(b)]$ using method of interpolation and hence find error approximation at x_0 and x_1.

(c) Derive the Newton-Cotes Quadrature formula $\int_a^b f(x)dx = \sum_{k=0}^{n} \lambda_k f(x_k)$ and deduce the Simpson's $\frac{1}{3}$ rule,

$$\int_a^b f(x)dx = \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)]$$

using method of interpolation.

(d) Derive the Newton-Cotes Quadrature formula $\int_a^b f(x)dx = \sum_{k=0}^{n} \lambda_k f(x_k)$ and deduce the Simpson's $\frac{3}{8}$ rule,

$$\int_a^b f(x)dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

using method of interpolation where $x_i = x_0 + ih$, $i = 1, 2, 3$ and $x_0 = a, x_3 = b$.

(2) (a) Derive error in Trapezoidal rule.

(b) Derive error in Simpson's $\frac{1}{3}$ rule.

(c) Derive error in Simpson's $\frac{3}{8}$ rule.

(3*) (Necessary and sufficient condition)

Let $I_n(f) = \sum_{j=0}^{n} w_{j,n} f(x_{j,n})$ $n \geq 1$, be a sequence of numerical integration formulas that approximate

$$I(f) = \int_a^b f(x)dx$$

. Let $\mathscr{F}$ be a family dense in $C[a, b]$. Then $I_n(f) \to I(f)$ all $f \in \mathscr{F}$ and

$$B = Sup\left\{\sum_{j-0}^{n} |w_{j,n}| : n \geq 1\right\} < \infty$$

(4) (a) Derive the trapezoidal rule from Newton-Cotes Quadrature formula $\int_a^b f(x)dx = \sum_{k=0}^{n} \lambda_k f(x_k)$ and deduce the composite trapezoidal rule

$$\int_a^b f(x)dx = \frac{h}{2}[f(x_0 + 2(f(x_1) + f(x_2) + \cdots + f(x_{n-1})) + f(x_n)]$$

(b) Derive the Simpson's rule from Newton-Cotes Quadrature formula $\displaystyle\int_a^b f(x)dx = \sum_{k=0}^{n} \lambda_k f(x_k)$ and deduce the composite Simpson's rule

$$\int_a^b f(x)dx = \frac{h}{3}[f(x_0+4(f(x_1)+f(x_3)+\cdots+f(x_{2n-1}))+2(f(x_2)+f(x_4)+\cdots+f(x_{2n-2}))+f(x_{2n}))+f(x_{2n})]$$

Number Theory 1

Quadratic Reciprocity

Objective Questions

Assume p and q are odd primes and ( ) denotes Legendre Symbol.

(1) The value of $(\frac{22}{7})$ is

(a) π (b) 1 (c)-1 (d) 0

(2) If $(\frac{2}{p})$=1, then

(a) p$\equiv 1\ or\ 7(mod\ 8)$ (b) p$\equiv 3\ or\ 5(mod\ 8)$

(c) p$\equiv 1\ or\ 7(mod 12)$ (d)p$\equiv 3\ or\ 5(mod\ 12)$

(3) If $(\frac{3}{p})$=1, then

(a)p$\equiv 1\ or\ 3(mod\ 12)$ (b) p$\equiv 1\ or\ 7(mod\ 12)$

(c) p$\equiv 1\ or\ 5(mod 12)$ (d)p$\equiv 1\ or\ 11(mod\ 12)$

(4) If $(\frac{-3}{p})$= 1, then

(a) p=6k+1 (b)p=6k+5 (c) p=8k+5 (d) cannot say

(5) If p=97 ,then

(a)$\left(\frac{-1}{p}\right) = 1\ and\ \left(\frac{2}{p}\right) = 1$ (b)$\left(\frac{-1}{p}\right) = -1\ and\ \left(\frac{2}{p}\right) = -1$

(c) $\left(\frac{-1}{p}\right) = -1\ and\ \left(\frac{2}{p}\right) = 1$ (d) $\left(\frac{-1}{p}\right) = 1\ and\ \left(\frac{2}{p}\right) = -1$

(6)Which of the following is correct:

(a)$\left(\frac{10}{83}\right) = 1\ and\ \left(\frac{10}{53}\right) = 1$ (b)$\left(\frac{10}{83}\right) = -1\ and\ \left(\frac{10}{53}\right) = -1$

(c) $\left(\frac{10}{83}\right) = 1\ and\ \left(\frac{10}{53}\right) = -1$ (d)$\left(\frac{10}{83}\right) = -1\ and\ \left(\frac{10}{53}\right) = 1$

(7) Which of the following is correct:

(a) $\left(\frac{15}{97}\right) = 1 \; and \; \left(\frac{15}{61}\right) = 1$ (b) $\left(\frac{15}{97}\right) = -1 \; and \; \left(\frac{15}{61}\right) = -1$

(c) $\left(\frac{15}{97}\right) = 1 \; and \; \left(\frac{15}{97}\right) = -1$ (d) $\left(\frac{15}{97}\right) = -1 \; and \; \left(\frac{15}{61}\right) = 1$

(8)if 1<a<p ,then the value of $\left(\frac{a^{p-1}}{p}\right) is$

(a) 1 (b) a (c) p-1 (d) p

(9) The value of $\left(\frac{2^{40}}{41}\right) is$

(a) 4 1 (b) 40 (c) 2 (d) 1

(10) The value of $\left(\frac{2^{5}}{41}\right) is$

(a) 5 (b) 2 (c) 1 (d) -1

(11) The value of $\left(\frac{(p-1)!}{p}\right) is$

(a) $(-1)^{\frac{p-1}{2}}$ (b) $(-1)^{\frac{p+1}{2}}$ (c)(p-1)! (d) None of these

(12) The value of $\sum_{j=1}^{p-1}(\frac{j}{p})$ is

(a) 1 (b) -1 (c) 0 (d)p-1

(13) In which of the following case, both congruence equations have solutions:

(a) $x^2 \equiv 3 \; mod \; 5, x^2 \equiv 5 \; mod \; 3$ (b) $x^2 \equiv 3 \; mod \; 7, x^2 \equiv 7 \; mod 3$

(c) $x^2 \equiv 5 \; mod \; 11, x^2 \equiv 11 \; mod \; 5$ (d) $x^2 \equiv 5 \; mod \; 13 \, , x^2 \equiv 13 \; mod \; 5$

(14) If g is primitive root of odd prime p then which of the following is true:

(a) g is quadratic residue of p (b)g is quadratic non residue of p

(c)$g^{p-1}$is quadratic non residue of p (d) g^{p-2} is quadratic residue of p

(15) The prime p for which $(\frac{10}{p})$=1 is

(a) p≡ 19(mod40) (b) p≡ 7(mod40) (c) p≡ 1(mod40) (d) p≡ 33 (mod40)

Number Theory Descriptive Questions 1

(1) Find all quadratic residues and quadratic non-residues for primes p=11,13,17,19.

(2) Given p=11 ,q=7.

(i)Consider residues of q,2q,…,($\frac{p-1}{2}$)q mod p. How many residues are greater than (p-1)/2?

(ii) Compute $\sum_{i=1}^{(\frac{(p-1)}{2}} \left[\frac{qi}{p}\right]$

(iii) Compute ($\frac{7}{11}$) in two ways using (i) and (ii).

(3) Evaluate:$\left(\frac{-23}{83}\right)$, $\left(\frac{-72}{131}\right)$, ($\frac{18}{43}$),($\frac{51}{71}$),($\frac{-35}{97}$).

(4) Determine whether following quadratic congruences are solvable:

(i) $x^2 \equiv$ 150(mod1009)

(ii) $x^2 \equiv$ 137(mod401)

(iii)$x^2 \equiv$ 73(mod173)

(iv) $x^2 \equiv$ 219(mod419)

(v) $x^2 \equiv -$43(mod79)

(5) If gis primitive root of p then prove that ($\frac{g}{p}$)=-1 .Also prove that the quadratic residues modulo p are congruent to g^2,g^4,g^6…..,g^{p-1} and quadratic non-residues are congruent to g ,g^3 ,g^5 … …..,g^{p-2}.

(6) If g is primitive root of p , prove that product of quadratic residues of p is congruent modulop to $g^{(p^2-1)/4}$and the product of quadratic non-residue modulo p to $g^{(p-1)^2/4}$.

(7) If p is an odd prime then prove that $\sum_{a=1}^{p-1}$($\frac{a}{p}$)=0.

(8) Find all primes p such that

(i) $\left(\frac{5}{p}\right) = 1$ $\quad (ii)\left(\frac{5}{p}\right) = -1$ $\quad (iii)\left(\frac{10}{p}\right) = 1$

(9) Find a prime number which is simultaneously expressible in the form $x^2 + y^2$,$u^2 + 2v^2$,$r^2 + 3s^2$.

(10) Let q=2p+1 .Show that ($\frac{p}{q}$) $= \left(\frac{-1}{p}\right)$.

(11) Let q be the least positive integer such that q<p and that $\left(\frac{q}{p}\right) = -1$, prove that q is prime.

(12) Let p$\equiv 3(mod 4)$ and q=2p+1. Then prove that q divides $M_p = 2^p - 1$.

(13) Let p=q+4a .Show that $\left(\frac{p}{q}\right) = \left(\frac{a}{q}\right) = \left(\frac{a}{p}\right)$.

(14) Show that (i) $\left(\frac{6}{p}\right) = 1$ if and only if p≡1,5,19 or 23(mod24).

(ii) $\left(\frac{7}{p}\right) = 1$ if and only if p$\equiv 1,3,9,19,25$ or $27(mod\ 28)$.

(15)Prove that if p>3 is an odd prime ,then $\left(\frac{-3}{p}\right) = 1\ if\ p \equiv 1(mod\ 6)$

$= -1$ if $p \equiv 5(mod\ 6)$

Hence show that the prime divisors p different from 3 of $n^2 - n + 1$ are of the form 6k+1.

(16) Show that there infinitely many primes of the form 6n+1.

(17) Solve the quadratic congruence $x^2 \equiv 11(mod\ 35)$.

(18) Prove that the odd prime divisors p of the integers $9^n + 1$are of the form p$\equiv 1(mod\ 4)$.

(19)For a prime p≡7(mod 8) , show that p|$2^{(p-1)/2} - 1$. Hence show that $2^n - 1$ are composite for n=11,23,83.

(20) Show that for any prime p$\equiv \pm 3(mod 8)$,the equation $x^2 - 2y^2$=p has no solution.

Number Theory

Practical 2

Jacobi Symbol and Quadratic congruences with composite modulii

Objective Questions

(1) If p=7 and q= 13, then

(a) $\left(\frac{-1}{pq}\right) = 1 \; and \; \left(\frac{2}{pq}\right) = -1$ (b) $\left(\frac{-1}{pq}\right) = -1 \; and \; \left(\frac{2}{pq}\right) = 1$

(c) $\left(\frac{-1}{pq}\right) = 1 \; and \; \left(\frac{2}{pq}\right) = 1$ (d) $\left(\frac{-1}{pq}\right) = -1 \; and \; \left(\frac{2}{pq}\right) = -1$

(2) Let a, b be positive integers which are relatively prime and b>1 be odd,then

(a) a is quadratic residue of b if and only if $(\frac{a}{b})$=1.

(b) If a is quadratic residue of b then $(\frac{a}{b})$=1 .

(c) If$(\frac{a}{b})$=1 ,thena is quadratic residue of b .

(d) None of these

(3) The congruence $x^2 \equiv a(mod 32)$ $(with\ 1 \leq a \leq 31)$is solvable for

(a) a = 1,9,17,25 only (b) a = 1,5,9,25 only

(c) a=1,5,9,21,25 only (d) a = 1,21,25 only

(4) Let p be an odd prime . The congruence $x^2 + \left(\frac{p+1}{4}\right) \equiv 0 \; mod \; p$

(a) Is solvable if p is of the type 4k+3 (b) Is not solvable if p is of the type 4k+3

(c) Is solvable if p is of the type 8k+7 (d) None of these

(5) Let p be a prime. There exist integers x ,y with (x,p)=1 ,(y ,p)=1 and $x^2 + y^2 \equiv 0 \; mod \; p$

(a) For all prime p (b) For all primes of the type 4k+3

(c) Only for p = 2 (d) For p=2 and primes of the type 4k+3

(6)The number of solutions of the congruence $x^2 \equiv 3 \; mod \; 11^2 23^2$ is

(a) 0 (b) 2 (c) 4 (d) 1

(7) The congruence $x^2 \equiv 231 \ mod \ 1105$ has

(a) 2 solutions (b) 1 solution (c) 4 solutions (d) no solutions

(8) The congruence $x^2 \equiv 25 \ mod \ 1013$ has

(a) 2 solutions (b) 1 solution (c) 4 solutions (d) no solutions

(9) Which of the following is correct ?

(a) The quadratic congruence $x^2 \equiv 12 \ mod \ 5$ has a solution.

(b) The quadratic congruence $x^2 \equiv 12 \ mod \ 7$ has a solution.

(c) The quadratic congruence $x^2 \equiv 12 \ mod \ 35$ has a solution.

(d) None of these.

(10) Which of the following is false?

(a) $x^2 \equiv a \ mod \ 2$always has a solution.

(b) $x^2 \equiv a \ mod \ 4$ has solution if and only if $a \equiv 1 \ mod \ 4$

(c)$x^2 \equiv a \ mod \ 2^n$,for n>2 has a solution if and only if $a \equiv 1 \ mod \ 8$

(d) None of (a),(b),(c) is true .

(11) If$x^2 \equiv a \ mod \ 2^n$,for n>2 has a solution then it has

(a) exactly 2 incongruent solutions (b) exactly 4 incongruent solutions

(c)exactly 1 solution (d) none of these

(12) The congruence $x^2 \equiv 19 \ mod \ 7^3$ has

(a) only one solution (b) two solutions (c) no solution (d) none of these

Number Theory

Descriptive Questions 2

(1) Evaluate $\left(\frac{21}{221}\right)$, $\left(\frac{215}{253}\right)$, $\left(\frac{631}{1099}\right)$.

(2) Which of the following congruences are solvable ?

(i) $x^2 \equiv 10 \; mod \; 127$

(ii) $x^2 \equiv 11 \; mod \; 61$

(iii) $x^2 \equiv 42 \; mod \; 97$

(iv) $x^2 \equiv 31 \; mod \; 103$

(3) Determine whether $x^2 \equiv 25 \; mod \; 1013$ is solvable.

(4) Determine whether $x^2 \equiv 231 \; mod \; 1105$ is solvable.

(5)Show that 7 and 18 are the only incongruent solutions of $x^2 \equiv -1 \; mod \; 5^2$

(6) Solve

(i)$x^2 \equiv 14 \; (mod \; 5^3)$

(ii)$x^2 \equiv 7 \; (mod \; 3^3)$

(iii)$x^2 \equiv 31 \; (mod \; 11^4)$

(7)Determine number of solutions of the congruence $x^2 \equiv 3 \; (mod \; 11^2 23^2)$ without actually finding them.

(8) Determine number of solutions of the congruence $x^2 \equiv 9 \; (mod \; 2^3 5^2 3)$ without actually finding them.

(9) Prove that if $x^2 \equiv a \; (mod \; 2^n)$, where a is odd and n≥ 3 has a solution, then it has exactly four incongruent solutions.

(10)Determine the values of a for which $x^2 \equiv a \; (mod \; 2^4)$ is solvable and find solutions.

Number Theory 3

Simple Finite Continued Fractions (SCF)

Objective Questions

Notation: For SCF [a_0 , a_1 , ……….., a_n] ; C_k = [a_0 , a_1 , ……….., a_k] = P_k / q_k ; $0 \leq k \leq n$

(1) The initial integer in the symbol [a_0 , a_1 , ……….., a_n] will be zero when the value of the fraction is

(a) $Positive$ & $Greater\ than\ one$ **(b)** $\boldsymbol{Positive}$ & $less\ than\ one$

$(C) Negaitive$ (d) Can not say

(2) The simple continued fraction (SCF) for $\frac{135}{79}$ is given by

(a) [0,1,1,2,2,3,3] (b) [2,1,2,1,3,3] **(c) [1,1,2,2,3,3]** (d) None of these

(3) The SCF for $-\frac{73}{116}$ is given by

(a) **[−1,2,1,2,3,4]** (b) [−2,1,1,2,3,4] (c) [−1,1,2,1,2,3] (d) None of these

(4) The SCF [0,1,2,3,4,3] represents

(a) $\frac{97}{135}$ **(b)** $\boldsymbol{\frac{97}{139}}$ (c) $\frac{34}{139}$ (d) None of these

(5) The SCF [−2,1,2,3,4,3] represents

(a) $-\frac{181}{139}$ (b) $-\boldsymbol{\frac{97}{139}}$ (c) $-\frac{34}{139}$ (d) None of these

(6) The SCF [2,1,2,1,2,1,2] equals

(a) [1,1,1,2,1,2,1,2] (b) [2,1,2,1,2,1,2,1] **(c) [2,1,2,1,2,1,1,1]** (d) None of these

(7) The SCF [2,1,2,1,2,2,1] equals

(a) [2,1,2,1,2,1,1,1] **(b) [2,1,2,1,2,3]** (c) [1,1,2,1,2,2,1] (d) None of these

(8) If r = [2,3,3,2] then $\frac{1}{r}$ is given by

(a) [2,3,3,2] (b) [½ , 1/3, 1/3, ½] **(c) [0, 2,3,3,2]** (d) None of these

(9) The value of the 4^{th} convergent of [2,3,1,4,2,3] is

(a) $\boldsymbol{\frac{95}{42}}$ (b) $\frac{43}{19}$ (c) 2 (d) None of these

(10) If u_6/u_5 represents quotient of two successive numbers in the Fibonacci Sequence then u_6/u_5 =

(a) [2,2,2,2,2] **(b) [1,1,1,1,1]** (c) [−1,1,1,1,1] (d) None of these

(11) Which of the following statement is correct :

(a)$p_0 = a_0; q_0 = 1$ (b) $p_0=1; q_0=1$ (c) $p_0=1; q_0= a_0$ (d) $p_0= a_1; q_0= a_0$

(12) Which of the following statement is correct :

(a)$p_1 = a_1; q_1 = 1$ (b) $p_1=1; q_1= a_1$ **(c) $p_1= a_1 a_0 + 1; q_1= a_1$** (d) $p_1=1; q_1= a_0$

(13) For $k \geq 1$ which of the following statement is correct :

(a) $p_k q_{k-1} - q_k p_{k-1} = (-1)^k$ **(b) $p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$**

(c) $p_k q_{k-1} - q_k p_{k-1} = (-1)^k a_k$ (d) $p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1} a_k$

(14) For $k \geq 2$ which of the following statement is correct :

(a) $p_k q_{k-2} - q_k p_{k-2} = (-1)^{k-2}$ (b) $p_k q_{k-2} - q_k p_{k-2} = (-1)^k$

(c) $p_k q_{k-2} - q_k p_{k-2} = (-1)^{k-2} a_k$ **(d) $p_k q_{k-2} - q_k p_{k-2} = (-1)^k a_k$**

(15) Which of the following statement is correct :

(a) $C_0 > C_2 > C_4 > C_6 \ldots\ldots$ (b) $C_1 < C_3 < C_5 < C_7 \ldots\ldots$ **(c) $C_0 < C_2 < C_4 < C_6 \ldots..$** (d) $C_1 \leq C_3 \leq C_5 \leq C_7 \ldots..$

(16) For a positive integer ‘c’ ; if SCF $[a_0, a_1, \ldots\ldots\ldots.., a_n] > [a_0, a_1, \ldots\ldots\ldots.., a_n + c]$ then

(a) n is odd (b) n is even (c) Both (a) & (b) (d) None of (a) & (b)

Number Theory Descriptive Questions 3

(1) Find SCF of 303/118 and Verify a) $p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}$ b) $p_n q_{n-2} - q_n p_{n-2} = (-1)^n a_n$

(2) Find SCF for the following : $\frac{57}{187}$; $-\frac{19}{51}$; $\frac{71}{55}$; $\frac{5!}{7}$; $\frac{3^4}{5}$; $\frac{p}{p+2}$ where p & p+2 are twin primes.

(3) Find rational numbers represented by following SCF:

a) [1,1,1,1,1,1,1] b) [2,2,1,1,2,1] c) [-2,1,3,5,7,9] d) [0,2,3,1,2,3] e) [1,1,1,2,2,2]

f) [-1,2,3,4,1] g) [2,2,2,2,2] h) [0,1,1,2,1,2,1,2]

(4) Find two representations of SCF for $\frac{6174}{1729}$ and hence or otherwise for $\frac{1729}{6174}$.

(5) For the following SCF , find C_k's & verify $C_0<C_2<C_4<C_6$….. and $C_1 > C_3 > C_5 > C_7$……

a) [−3,2,4,1,1,3,2] b) [0,4,,3,5,8,2,1,7]

(6) Let fn be the n^{th} Fibonacci number. Find SCF for f_{n+1}/f_n. Prove that $f_n^2 - f_{n+1}.f_{n-1} = (-1)^{n-1}$ for all $n \geq 2$

(7) Let Pell number P_n be defined as follows : $P_0 = 0$, $P_1 = 1$ & $P_n = 2P_{n-1} + P_{n-2}$ $\forall$ $n \geq 2$. If $t \in Q$ is such that it's SCF consists of n, 2's then prove that $t = \frac{p_{n+1}}{p_n}$

(8) If $C_k = \frac{p_k}{q_k}$ is the k^{th} convergent of the simple continued fraction [a_0 , a_1 , ………..,a_n] then prove that $q_k \geq 2^{(k-1)/2}$ for $2 \leq k \leq n$

(9) Find the SCF representation of 3·1416 ; 3·14159

(10) If $C_k = \frac{p_k}{q_k}$ is the k^{th} convergent of the simple continued fraction [a_0 , a_1 , ………..,a_n] and $a_0 > 0$ then show that $\frac{p_k}{p_{k-1}} =$ [a_k , a_{k-1} , ………..,a_1, a_0]

(11) Using SCF of suitable rational solve the following :

a) 118x + 303y =1
b) 18x +5y = 18
c) 158x – 57y =1

Number Theory

Practical 4

Simple Infinite Continued Fractions (SICF)

Objective Questions

(1) The SICF of $\sqrt{15}$ is given by

(a) $[3,\overline{1,3}\,]$ **(b) $[3,\overline{1,6}\,]$** (c) [3,1,2,3,4,8,5] (d) None of these

(2) The SICF of $\sqrt{2} - 1$ is given by

(a) $[\,0,\overline{1,2}\,]$ (b) $[\,\overline{1,3}\,]$. **(c) $[\,0,\overline{2}\,]$** (d) None of these

(3) If $\alpha = [\overline{2,1}\,]$ then α equals

(a) $1 + \sqrt{3}$ (b) 2½ (c) $1 - \sqrt{3}$ (d) None of these

(4) The SICF of $\frac{(e-1)}{(e+1)}$ is given by

(a) [e,1,e, − 1] **(b) [0,2,6,10,14,18,……..]** (c) [2,1,2,1,4,1,8,…..] (d) None of these

(5) The SICF of $\frac{(e^2-1)}{(e^2+1)}$ is given by

(a) **[0,1,3,5,7,9,……]** (b) [0,2,6,10,14,18,……..]

(c) [1,1,4,5,7,8……..] (d) None of these

(6) The SICF [1,1,1,1,……] represents

(a) 1 (b) 1.1111 **(c) $\frac{1+\sqrt{5}}{2}$** (d) None of these

(7) For n Є IN , $\sqrt{n^2 + 1}$ =

(a) $[\,n,\overline{n,2n}\,]$ (b) **$[\,n,\overline{2n}\,]$** (c) $[\,n,\overline{1,2n}\,]$ (d) None of these

(8) Let x = [1,3,1,5,1,7,1,9,……]. If Cn = $\frac{p_n}{q_n}$ is the n^{th} convergent of x then we know that $|\,x - Cn| < \frac{1}{q_n\, q_{n+1}}$, using this inequality the rational approximation to x correct upto 3 decimal places is

(a) $\frac{34}{27}$ (b) $\frac{301}{239}$ (c) $\frac{267}{212}$ (d) None of these

(9) If x = $\frac{1+\sqrt{13}}{2}$ then x =

(a) $[2,\overline{3}\ ]$ (b) $[2,\overline{1,3}\ ]$ (c) $[2,1,\overline{3}\ ]$. (d) None of these.

(10) If $\alpha =$ [a_0 , a_1 , a_2……….,] and C_n = [a_0 , a_1 , a_2……….., a_n] is the n^{th} convergent then $\alpha =$

(a) $\lim_{n\to\infty} C_n$ (b) $\lim_{n\to\infty} C_{n-1}$ (c) Both (a) and (b) (d) None of these.

(11) If $C_k = \frac{p_k}{q_k}$ is the k^{th} convergent of SICF [a_0 , a_1 , a_2 ……….,] then

(a) (C_{10} ,C_{11}) $\subseteq$ (C_2 ,C_3) **(b) (C_{10} ,C_{11}) $\subseteq$ (C_{12} ,C_3)**

(c) (C_{10} ,C_{11}) $\subseteq$ (C_8 ,C_3) (d) none of these

Number Theory

Descriptive Questions 4

(1) Obtain SICF for the following :

a) $\sqrt{2}$ b) $\sqrt{3}-1$ c) $\sqrt{2}+1$ d) $2-\sqrt{3}$ e) $\sqrt{11}$ f) $\sqrt{22}$ g) $\sqrt{41}$ h) $\frac{1}{\sqrt{2}}$

$i)$ $\frac{1+\sqrt{7}}{2}$ J) $\frac{5+\sqrt{37}}{4}$

2) Obtain SICF for the following :

a) $\sqrt{\frac{22}{7}}$ b) $\sqrt{\frac{3}{2}}$ C) $\sqrt{\frac{5}{3}}$ d) $\sqrt{\frac{8}{5}}$

3) Find value of the following SICF :

(a) [0,1,1,1,1,1,……] **(**b) [2,1,2,1,,……..] (c) [1,2,1,2,……..] e) [$1,\overline{1,2}$]
f) [$\overline{1,1,2}$] g) [$1,2,\overline{2,3}$]

4) Assume e = 2.718281828 and find first four terms of SICF for $\frac{(e-1)}{(e+1)}$ and $\frac{(e^2-1)}{(e^2+1)}$

5) SICF for ‘e’ is given by [2,1,2,1,1,4,1,1,4,…..] ; find e correct upto 4 decimal places.

6) Prove $\sqrt{n^2+1}$ = [n, $\overline{2n}$] for all n $\in IN$ hence find SICF for $\sqrt{17}$

7) Let x= [1,2,3,…..]. Find least n such that n^{th} convergent $\frac{p_n}{q_n}$ approximates x correct upto 5 decimal places.

8) Prove that $\sqrt{(4m^2+4)}$ = [2m,$\overline{m,4m}$]

9) Assume that π = [3,7,15,1,212,…..]

a) Find first 5 convergents.
b) Show $\frac{355}{113}$ approximates π correct upto 6 decimal places
c) Obtain rational $\frac{a}{b}$ such that $|\pi - \frac{a}{b}| < \frac{1}{\sqrt{5}\,b^2}$

Number Theory : PRACTICAL 5

NUMBER-THEORETIC FUNCTIONS

Objective Questions

(1) Which of the following is the solution of $\tau(n) = 4$?
(a)2 (b) 4 (c) 8 (d) 16

(2) Which of the following is the solution of $\sigma(n) = 4$?

(a) 1 (b) 2 (c) 3 (d) 4

(3)$\sigma(n) = 2^k, k \in \mathbb{N}$ has no solution for

(a) k=1 (2) k=2 (c) k=3 (d) k=5

(4) If the prime $p \equiv -1(mod\ 4) and\ if\ 2|k,$ then $\sigma(p^k)$ is congruent 4 to

(a) 0(mod 4) (b) 1(mod 4) (c) 2(mod 4) (d) cannot say

(5) Which of the following is not perfect?

(a)$2^2(2^3 - 1)$ (b)$2^4(2^5 - 1)$ (c)$2^6(2^7 - 1)$ (d)$2^{10}(2^{11} - 1)$

(6)Let p and q be distinct primes and $n = pq$. Then $\sigma(n)\ is$

(a)pq (b)$(p + 1)(q + 1)$ (c) n+1 (d) $None$

(7) Let p and q be distinct primes and $n = pq$. Then $\tau(n)\ is$

(a)1(b)2(c) 4 (d)$p + q$

(8) If $2^n - 1 \quad and \quad 2^n + 1$ are both primes for $n \in \mathbb{N}$,then

(a) n must be odd (b) there are infinitely many such n

(c) n=2 (d) None of the above

(9) Let $F_n = 2^{2^n} + 1, then$ for $n \neq m, \gcd(F_n, F_m)\ is$

(a) n (b) 2^m (c) 1 (d) None

(10) Let p and q be distinct primes and $M_n = 2^n - 1.\ Then$ gcd $(M_p\ , M_q)$ is

(a) 1 (b) p (c) q (d) n

.

NUMBER THEORY DESCRIPTIVE QUESTIONS-5

1. Prove that $\prod_{d|n} d$=$n^{\frac{\tau(n)}{2}}$
2. Prove that if n is a natural number such that $\tau(n) = q, where\ q\ is\ prime$, then n=$q^{p-1}$for some prime p.
3. Find the least integer n such that $\tau(n) = 25$.
4. If $\omega(n\ )$ denotes the number of distinct prime factors of n, prove that $\tau(n) \geq 2^{\omega(n)}$.
5. Prove that $\tau(n)$ is odd if and only if n is a square.
6. If $\tau(n) = 4$, what can be said about the canonical factorization of n?
7. Prove that if $\sigma(n)$ is prime , then $n = p^k$, where p is prime and $k \geq 1$.
8. Prove that if $\sigma(p^k) = n$, where p is prime, then p$|(n-1)$.
9. Prove that if n is odd, then $\tau(n) \equiv \sigma(n)(mod\ 2)$.
10. Prove that if p is prime and $n \geq 2$, then $\sigma(p^{n^2-1})$ is composite.
11. Prove that if n ≡7(mod 8), then $\sigma(n)$ ≡0(mod 8)
12. Prove that if n ≡23(mod 24), then $\sigma(n)$ ≡0(mod 24)
13. Prove that if p and q are distinct primes such that $\sigma(p^2) = \sigma(q^4)$, then p=5 and q=2.
14. Prove that there are no primes p and q such that $\sigma(p^2) = \sigma(q^6)$.
15. Prove that $\sum_{d|n} |\mu(d)| = 2^{\omega(n)}$
16. Prove that if p is prime and $p \equiv 1(mod\ 3)$, then the equation $\varphi(x) = 2p$ has no solution.
17. Prove that $\varphi(n)|n\ \ if\ and\ only\ if\ \ n = 1,\ 2^\alpha, or\ 2^\alpha 3^\beta$ where α, β are natural numbers.
18. Prove that if $n \not\equiv 2(mod\ 4), then\ \mu)(n^3 - n) = 0$.
19. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots . p_r^{\alpha_r}$ be the prime factorization of the integer n>1. If f is a multiplicative function that is not identically zero, prove that

$$\sum_{d|n} \mu(d) f(d) = (1 - f(p_1))(1 - f(p_2)) \dots .. (1 - f(p_r)\ \ )$$

20, If n is a perfect number prove that If the integer n>1 has the prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots . p_r^{\alpha_r}$, establish the following:

(a) $\sum_{d|n} \mu(d)\tau(d) = (-1)^r$
(b) $\sum_{d|n} \mu(d)\sigma(d) = \ (-1)^r p_1 p_2 \dots \dots . p_r$
(c) $\sum_{d|n} \frac{\mu(d)}{d}\ = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \dots .. (1 - \frac{1}{p_r})$
(d) $\sum_{d|n}\ d\mu(d) = (1 - p_1)(1 - p_2) \dots \dots . (1 - p_r)$

21. Verify each of the statements below:

(a) No power of a prime can be a perfect number.

(b) A perfect square cannot be a perfect number.

(c) The product of two odd primes is never a perfect number.

22. If n is a perfect number , prove that $\sum_{d|n}\frac{1}{d} = 2$

23. If the three numbers $p = 3.2^{n-1} - 1, q = 3.2^{n} - 1\ and\ r = 9.2^{2n-1} - 1$ are all prime and $n \geq 2, then\ show\ that\ 2^{n}pq\ and\ 2^{n}r$ are amicable numbers.

24. Prove that the Mersenne number M_{13} is a prime, hence the integer $n = 2^{12}(2^{13} - 1) is\ perfect.$

25. For $n \geq 2$, show that the last digit of the Fermat number $F_n = 2^{2^n} + 1$ is 7.

26. Show that the Fermat number $F_n = 2^{2^n} + 1$ is never a perfect square.

27. For n>0, number $F_n = 2^{2^n} + 1\ is$ never a triangular number.

28. Show that every Fermat number, F_n is a prime or a pseudoprime.

………

Number Theory(practical no.6)

Objective Questions

(1) The fundamental solution of $x^2 - 3y^2 = 1$ is

(a) $(2, 1)$ (b) (7,4) (c) (1 ,0) (d)None

(2) Pell's equation $x^2 - 13y^2 = -1$ has

(a) Only one solution (b) no solution

(c) infinitely many solutions (d) None

(3) Let $\sqrt{29} = [a_0; \overline{a_1, a_2, \ldots\ldots a_n}]$. Then $(a_n - a_{n-1} + a_{n-2} - \cdots \ldots\ldots + a_2 - a_1)$ equals

(a) 29 (b) -1 (c) 3 (d) None

(4) Pell's equation $x^2 - 30y^2 =$ -1 has

(a) only one solution (b) no solution (c)infinitely many solutions (d) None

(5) If $\alpha = 1 + \sqrt{2}$, $\beta = 1 - \sqrt{2}$ $then\ the\ Pell\ numbers$ P_n and Q_n are given as

(a) $P_n = \frac{(\alpha^n + \beta^n)}{2\sqrt{2}}$, $Q_n = \frac{(\alpha^n - \beta^n)}{2}$ (b) $Q_n = \frac{(\alpha^n + \beta^n)}{2\sqrt{2}}$, $P_n = \frac{(\alpha^n - \beta^n)}{2}$

(c) $P_n = \frac{(\alpha^n - \beta^n)}{2\sqrt{2}}$,$Q_n = \frac{(\alpha^n + \beta^n)}{2}$ (d) None

(6) $For\ the\ Pell\ numbers\ P_n\ and\ Q_n with\ n \geq 1,\ Q_n - P_n =$

(a) P_{n-1} (b) Q_{n-1} (c) $2P_{n-1}$ (d) None

(7) If x_1, y_1 is a fundamental solution of $x^2 - dy^2 = 1$, then every positive solution of the equation is given by x_n, y_n which satisfy

(a) $y_n + x_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$ (b) $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$

(c) $Both\ (a)\ and\ (b)$ (d) $None$

(8) if $C_n = \frac{p_n}{q_n}$ (n=0,1,2,.....) is the n^{th} convergent and k is the length of the period

Of the infinite SCF of $\sqrt{d}$, then p_{nk-1} , q_{nk-1} is a solution of $x^2 - dy^2 = 1$

(a) k=2, n=5 (b) k=3, n=4 (c) Both (a) and (b) (d)None

(9) Every Carmichael number, an absolute pseudoprime

(a) is odd (b) has atleast 3 distinct prime factors (c) composite

(d) (a), (b),and (c)

(10) If d is divisible by a prime $p \equiv 3(mod\ 4)$, then the equation $x^2 - dy^2 = -1$ has

(a) no solution (b) infinitely many solutions (c) one solution (d) None

………

DESCRIPTIVE QUESTIONS

1) Establish that if $x_0\ , y_0\ \textit{is a solution of}\quad x^2 - dy^2 = -1$, then $x = 2dy_0{}^2 - 1,\quad y = 2x_0y_0$ satisfies $x^2 - dy^2 = 1.$

2) Find the fundamental solutions of
   i) $x^2 - 3y^2 = 1$
   ii) $x^2 - 41y^2 = 1$
   iii) $x^2 - 6y^2 = 1$
   iv) $x^2 - 47y^2 = 1$

3) Prove solutions (x_n, y_n) of $x^2 - 6y^2 = 1$ are given by $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 2 & 5 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

4) Prove that if $(x_1\ , y_1\ )$ is fundamental solution of $x^2 - dy^2 = 1$, then all solutions are given by $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & dy_1 \\ y_1 & x_1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

5) Does $x^2 - 15y^2 = -1$ have any solution? Explain using

   (i) continued fractions (ii) quadratic residues

6) Prove that if $(x_1\ , y_1\ )$ is fundamental solution to the associated Pell's equation $x^2 - dy^2 = -1$, then all solutions are given by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & dy_1 \\ y_1 & x_1 \end{bmatrix}^{2n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

………

Number Theory Practical 7

Miscellaneous Theory Questions

Unit 1

(1) Define the terms Quadratic residue and Quadratic non-residue of an odd prime p. If p is an odd prime and gcd (a,p)=1 then prove that a is quadratic residue of p if and only if $a^{\frac{p-1}{2}} \equiv 1(mod\,p)$ and a is quadratic non-residue of p if and only if $a^{\frac{p-1}{2}} \equiv -1(mod\,p)$.

(2) Define Legendre symbol $(\frac{a}{p})$. If p is an odd prime and a and b are relatively prime to p then prove that

(i) $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}}\, mod\ p.$

(ii) $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$

(iii) a$\equiv b\ mod\ p$ implies $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.

(iv) If (a,p)=1 then $\left(\frac{a^2}{p}\right) = 1$ and $\left(\frac{a^2 b}{p}\right) = \left(\frac{b}{p}\right)$.

(v) $\left(\frac{1}{p}\right) =1$ and $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$.

(3) Show that $\left(\frac{-1}{p}\right) = 1$ if p$\equiv 1\ mod\ 4$ and $\left(\frac{-1}{p}\right) = -1$ if p$\equiv 3\ mod\ 4$.

Hence Show that there are infinitely many primes of the form 4k+1.

(4) State and prove Gauss' Lemma.

(5) If p is an odd prime and (a,2p) = 1 , then show that $\left(\frac{a}{p}\right) = (-1)^t$, where $t = \sum_{j=1}^{(p-1)/2} \left[\frac{ja}{p}\right]$

and $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

(6) If p is an odd prime ,then prove that (i) (2/p)= 1 if p$\equiv 1\ or\ 7\ mod\ 8$ and

(ii) (2/p) = -1 if p$\equiv 3\ or\ 5\ mod\ 8$

(7) State and prove Quadratic Reciprocity Law.

(8) If p is an odd prime ,then prove that (i) (3/p)= 1 if p$\equiv 1\ or\ 11\ mod\ 12$ and

(ii) (3/p) = -1 if p$\equiv 5\ or\ 7\ mod\ 12$

(9)) If p is an odd prime ,then prove that(i) (-2/p)= 1 if p$\equiv 1$ or 3 mod 8 and

(ii) (-2/p) = -1 if p$\equiv 5$ or 7 mod 8

(10) Define Jacobi Symbol $(\frac{P}{Q})$ for Q positive and odd. If Q and Q' are odd and positive then show that

(i) $(\frac{P}{Q})$ $(\frac{P}{Q'})$ =$(\frac{P}{QQ'})$

(ii) $(\frac{P}{Q})$ $(\frac{P'}{Q})$ = $(\frac{PP'}{Q})$

(iii) (P ,Q) = 1 then $(\frac{P^2}{Q})$ = $(\frac{P}{Q^2})$ = 1

(iv) If (PP' ,QQ') = 1 ,then $(\frac{P^2P'}{Q^2Q'})$ = $(\frac{P}{Q})$

(v) P'$\equiv P$ mod Q implies $(\frac{P'}{Q})$ = $(\frac{P}{Q})$

(11) If Q is odd and Q>0 , then show that $(\frac{-1}{Q})$ = $(-1)^{\frac{Q-1}{2}}$ and $\left(\frac{2}{Q}\right)$ = $(-1)^{(Q^2-1)/8}$.

(12) State and prove Generalized Quadratic Reciprocity Law.

(13) If p is an odd prime and (a , p) = 1, then show that the congruence

$x^2 \equiv a\ (mod\ p^n)\ \ n \geq 1$ has a solution if and only if $\left(\frac{a}{p}\right) = 1.$

(14) If a is an odd integer . Then show that

(a) $x^2 \equiv a\ (mod\ 2)$ always has a solution.

(b) $x^2 \equiv a\ (mod\ 4)$ has a solution if and only if a$\equiv 1\ (mod\ 4)$

(c) $x^2 \equiv a\ (mod\ 2^n)$ for $n \geq 3$, $has\ a\ solution\ if\ and\ only\ if\ a \equiv 1 (mod\ 8)$

(15) If n = $2^{k_0}{p_1}^{k_1} \dots \dots ..{p_r}^{k_r}$ is the prime factorization of n>1 and (a,n)=1 , then show that $x^2 \equiv a\ (mod\ n)$ is solvable if and only if

(a) $\left(\frac{a}{p_i}\right) = 1\ for\ i = 1,2, \dots r$

(b) a $\equiv 1\ (mod\ 4)$ if 4|n ,but 8 does not divide n

(c) a$\equiv 1\ (mod\ 8)$ if 8|n.

Unit II

Continued Fraction

Notation: For SCF $[a_0, a_1, \ldots\ldots\ldots, a_n]$; $C_k = [a_0, a_1, \ldots\ldots\ldots, a_k] = P_k/q_k$; $0 \le k \le n$

1) Show that any rational number can be written as a finite simple continued fraction and every finite SCF represents a rational number.
2) For $0 \le k \le n$, define P_k and q_k as $p_0 = a_0; q_0 = 1$; $p_1 = a_1a_0 + 1$; $q_1 = a_1$ and $r_n = [a_0, a_1, \ldots\ldots\ldots, a_n] = P_n/q_n$ then prove that
   a)For $k \ge 2$
   $\frac{p_k}{q_k} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}$
   b) $P_k q_{k-1} - P_{k-1} q_k = (-1)^{k-1}$ For $k \ge 1$
   c) $P_k q_{k-2} - P_{k-2} q_k = (-1)^k a_k$ For $k \ge 2$
   d) $\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{(-1)^{K-1}}{q_k q_{k-1}}$
   e) If n is odd then $r_n < r_{n-2}$ and if n is even then $r_{n-2} < r_n$
3) Prove the following:
   a) The convergents with even subscripts form a strictly increasing sequence.
   b) The convergents with odd subscripts form a strictly decreasing sequence.
   c) Every convergent with an odd subscript is greater than every convergent with an even subscript .
4) If gcd(a,b) =1 and $a/b = [a_0, a_1, \ldots\ldots\ldots, a_n]$ where n is odd. Then prove that $x = cq_{n-1}$ & $y = -cp_{n-1}$ gives a solution of $ax+by = c$
5) Prove that every Simple Infinite Continued fraction (SICF) represents an irrational number and conversely.
6) Prove that two distinct Simple Infinite Continued fractions converge to different numbers.
7) If $1 \le b \le q_n$ then prove that the rational number a/b satisfies $|x - \frac{p_n}{q_n}| \le |x - \frac{a}{b}|$
8) State and prove Dirichlet's Theorem about quadratic approximation.

Unit III

$(M_p is\ Mersenne\ number, F_n is\ Fermat\ number)$

1. Define the $M\ddot{o}bius$ μ-function. Show that it is a multiplicative function.

2. State and prove the $Mobius$ inversion formula.

3. If 2^k-1 is prime (k >1), then show that $n = 2^{k-1}(2^k - 1)$ is perfect and every even perfect number is of this form.

4. If p and q=2p+1 are primes, then prove that either q$|M_p$ or q$|M_p + 2$, but not both.

5. If q=2n+1 is prime then establish the following,
   (a) q$|M_n$ provided that $q \equiv 1(mod\ 8)\ or\ q \equiv 7(mod\ 8)$
   (b) q$|M_n + 2$ provided that $q \equiv 3(mod\ 8)\ or\ q \equiv 5(mod\ 8)$

6. If p is an odd prime then show that any odd divisor of M_p is of the form 2kp+1.
7. If p is an odd prime then show that any odd divisor q of M_p is of the form $q \equiv \pm 1(mod\ 8)$.

8. State and prove Korselt's criterion for Carmichael numbers.

9. Prove that every Carmichael number is the product of three or more distinct odd factors. Furthermore , prove that, if n is a Carmichael number and if p is an odd prime , then p divides n if and only if p-1|n-1.

10. Show that any absolute pseudoprime is square free.

11. Let x_1, y_1 be the fundamental solution of $x^2 - dy^2 = 1$. Then show that every pair of integers x_n, y_n defined by the condition
$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n \qquad \text{n=1,2,3.........}$$
is also a positive solution.

12. If x_1, y_1 is the fundamental solution of $x^2 - dy^2 = 1$, then prove that every positive solution of the equation is given by x_n, y_n where x_n *and* y_n are the integers determined from

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n \qquad n=1,2,3\ldots\ldots$$

13. Let d be a positive nonsquare integer. Let (x_0, y_0)= (1,0) be the trivial solution to the equation $x^2 - dy^2 = 1$.Then show that all solutions to the equation in non-negative integers may be expressed by each of the following forms, where the integer $n \geq 1$.

a) $$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & dy_1 \\ y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}$$

b) $$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & dy_1 \\ y_1 & x_1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

14) Assuming that equation $x^2 - dy^2 = -1$ is solvable, let x_1, y_1 be the smallest positive solution. Prove that all solutions of equation

$x^2 - dy^2 = -1$ are given by x_n, y_n where

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n \qquad \text{with } n=1,3,5,7\ldots..$$

and that all solutions of $x^2 - dy^2 = 1$ are given by x_n, y_n

with n=2,4,6,8…….

………